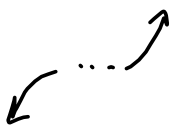
Writing Project #3

We will be working with $f(x) = 9x^5 - 69x^4 + 130x^3 + 6x^2 - 256x - 120$ for most of this test. We'll say everything about this polynomial that's worth saying.

1. (2 pts) Describe the end behavior of f with a simple graphic.

$9x^5$



Book: $\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

2. (2 pts) Use Descartes' Rule of Signs to determine the *possible* number of positive and negative zeros.

$$f(x) = \underbrace{9x^5 - 69x^4}_{1} \underbrace{+130x^3}_{2} \underbrace{+6x^2 - 256x^1}_{3} - 120$$

$\boxed{\text{3 or 1 positive zeros.}}$

$$f(-x) = -9x^5 - 69x^4 \underbrace{-130x^3 + 6x^2}_{1} \underbrace{+ 256x - 120}_{2}$$

$\boxed{\text{So, 2 or 0 negative zeros}}$

3. (2 pts) Use the Rational Zeros Theorem to determine the *possible* rational zeros (roots) of f.

$$f(x) = 9x^5 - 69x^4 + 130x^3 + 6x^2 - 256x - 120$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$,
where the coefficients a_k are integers.

If $f\left(\frac{p}{q}\right) = 0$, then p is a product of factors of a_0 & q is a product of factors of a_n

$9 = 3 \cdot 3$

$9x^5$ $\quad a_5 = 9 = a_n$

-120 $\quad a_0 = -120$

$a_5 = 9 = 3 \cdot 3$ Denominators

$a_0 = -120$ ugh!

$= -2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$

Numerators

Scratch
$$\begin{array}{r|l} 2 & 120 \\ 2 & 60 \\ 2 & 30 \\ 3 & 15 \\ & 5 \end{array}$$

Looks good!

$\pm 1, \pm \frac{1}{3}, \pm \frac{1}{9}, \pm 2, \pm \frac{2}{3}, \pm \frac{2}{9}, \pm 4, \pm \frac{4}{3}, \pm \frac{4}{9}, \pm 8, \pm \frac{8}{3}, \pm \frac{8}{9},$

$\pm 3, \cancel{\pm \frac{3}{3}}, \cancel{\pm \frac{3}{9}}, \pm 6, \cancel{\pm \frac{6}{3}}, \cancel{\pm \frac{6}{9}}, \pm 12, \cancel{\pm \frac{12}{3}}, \cancel{\pm \frac{12}{9}}$

$\pm 24, \cancel{\pm \frac{24}{3}}, \cancel{\pm \frac{24}{9}}, \pm 5, \pm \frac{5}{3}, \pm \frac{5}{9}$? $, \pm 10, \pm \frac{10}{3}, \pm \frac{10}{9},$

$\pm 20, \pm \frac{20}{3}, \pm \frac{20}{9}, \pm 40, \pm \frac{40}{3}, \pm \frac{40}{9}, \pm 15, \cancel{\pm \frac{15}{3}}, \cancel{\pm \frac{15}{9}} = \frac{5}{3},$

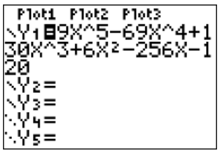
$\pm 30, \cancel{\pm \frac{30}{3}}, \cancel{\pm \frac{30}{9}}, \pm 60, \cancel{\pm \frac{60}{3}}, \cancel{\pm \frac{60}{9}}, \pm 120, \cancel{\pm \frac{120}{3}}, \cancel{\pm \frac{120}{9}}$

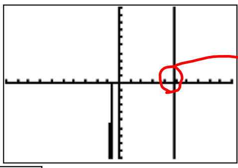
4. (2 pts) Using the information, above, find all real zeros of f. Finding all zeros includes finding the multiplicity of each. This means performing multiple synthetic divisions. Always check for multiplicity greater than 1 with another synthetic division, just in case.

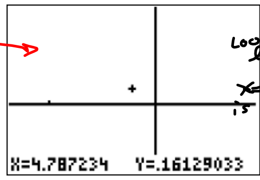
I recommend using a graphing calculator/grapher to reduce the guesswork. But if you rely on digital approximations, things won't factor, properly, or at least they might not.

$$f(x) = 9x^5 - 69x^4 + 130x^3 + 6x^2 - 256x - 120$$

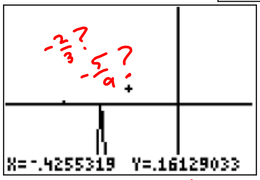
$[-10,10] \times [-10,10]$ $(0,-120) \to$ y-int.







Looks like x=5 is zero of f.



$-\frac{2}{3}$? $-\frac{5}{9}$?

$(x-5)(9x^4 - 24x^3 + 10x^2 + 56x + 24)$

Depressed Polynomial. Not divisible by 5

Check $x=5$: $f(5)=0$?

5	9	-69	130	6	-256	-120
		45	-120	50	280	120
5	9	-24	10	56	24	0 Sweet!
		45	105	575	Huge	
	9	21	115	Big	enormous	$\neq 0$

$\frac{1150}{2} = 575$

5 is a zero of multiplicity 1.

Now we're working with the depressed polynomial, Try $-\frac{2}{3}$

$9x^4 - 24x^3 + 10x^2 + 56x + 24$

Divide by $x + \frac{2}{3}$:

$(x-5)(x+\frac{2}{3})(9x^3 - 30x^2 + 30x + 36)$

$-\frac{2}{3}$	9	-24	10	56	24
		-6	20	-20	-24
$-\frac{2}{3}$	9	-30	30	36	0 sweet!
		-6	24	-36	
	9	-36	54	0 sweet!	

Now we have $(x-5)(x+\frac{2}{3})^2(9x^2 - 36x + 54)$

$9x^2 - 36x + 54$ is our current depressed poly

Clobber it with Quadratic Formula

$a=9, b=-36, c=54$

has same zeros as

$9(x^2 - 4x + 6)$, so

$a=1, b=-4, c=6$

$b^2 - 4ac = (-4)^2 - 4(1)(6)$

$= 16 - 24 = \boxed{-8 = b^2 - 4ac}$

No real zeros

$9x^2 - 36x + 54$ is irreducible over the real numbers.

$\therefore$ #4 sol'n is

$$\boxed{\begin{array}{l} x = -\frac{2}{3} \text{ w/ mult} = 2 \\ x = 5 \quad " \quad " = 1 \end{array}}$$

5. (2 pts) From your work, above, factor f over the real numbers. This will involve an irreducible quadratic factor that your grapher has no way of helping you to see. without the synthetic divisions in #4, bringing

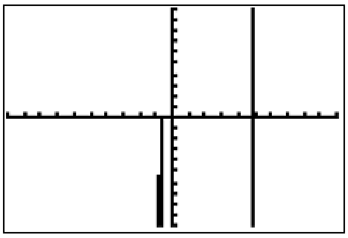
Simply write the factored form I already showed you!

Maybe I should just make #4 a 2-parter!

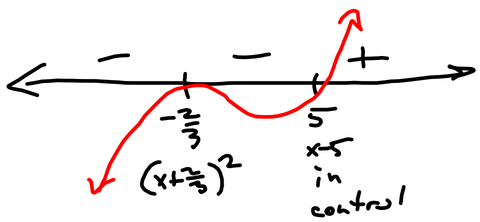
$$\boxed{f(x)=(x-5)(x+\tfrac{2}{3})^2(9x^2-36x+54)}$$

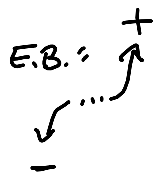
6. (2 pts) Give a rough sketch of f from all of the above information. This is an *art* whose essence is really only found in my videos. If you're too tied to your grapher's output, you'll not capture the real essence of what's going on, or the key features I'm always looking for. Your picture will be more "vertical" than it should be.

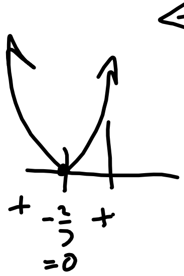
Standard Window

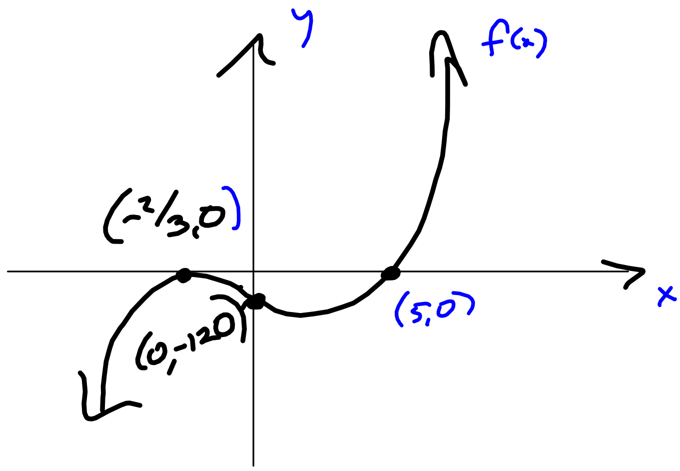


$(x-5)(x+\frac{2}{3})^2(9x^2-36x+54)$









7. (2 pts) Now we've covered everything *real* about f. Let's use that work to find *all* the roots of f and *split f into linear factors*. 5 roots (counting repetitions) are *guaranteed by the Fundamental Theorem of Algebra,* and we have found the 3 real ones. The other 2 are nonreal, hiding inside the irreducible quadratic polynomial that remains as the last, very very depressed piece that's not broken all the way down in #5.

Now do your quadratic equation thing to find the 2 nonreal roots. Finally, apply the Factor Theorem to *all* the above work, and represent f as a product of linear factors, $f(x)=a(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_w)^{m_w}$.
Don't forget the leading coefficient, a.
This wrings (almost) every useful drop of the Theorems on Polynomials out of f, so now on to Rational Functions, which are *quotients* of polynomials!

$f(x) = (x-5)(x+\frac{2}{3})^2(9x^2-36x+54)$

$9x^2-36x+54=0$

$\Rightarrow x^2-4x+6=0$, by previous work on discriminant in #4

$b^2-4ac=-8$

$\rightsquigarrow \sqrt{-8}$:

2 | 8
2 | 4
2

$2\sqrt{2}$

$x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{4\pm 2\sqrt{2}}{2(1)} = \dfrac{2(2\pm\sqrt{2})}{2} = 2\pm\sqrt{2}$

what about the -8?

oh yeah. $2\pm i\sqrt{2}$

So $\boxed{x = 2\pm i\sqrt{2}}$

from $\sqrt{-8} = i\sqrt{8} = i(2\sqrt{2})$
$= 2i\sqrt{2}$

$f(x) = (x-5)(x+\frac{2}{3})^2(x-(2+i\sqrt{2}))(x-(2+i\sqrt{2})) = x^5 + \ldots$

Almost
Need that "9," the leading coefficient,

$\boxed{f(x) = 9(x-5)(x+\frac{2}{3})^2(x-(2+i\sqrt{2}))(x-(2+i\sqrt{2}))} = 9x^5+\ldots$ what we want!

8. (5 pts) Sketch the graph of $R(x)=\dfrac{6x^2+x-77}{x^2+2x-15}$, showing all intercepts, asymptotes, and capturing the *essential features* of the shape of the graph. If you're a slave to your grapher, and oblivious to the features I'm looking for, it'll jump off the page at me (and be bad).

Note: There *is* a subtle feature to this graph that I downplay on tests, but you should pick up on with a take-home, namely, the horizontal asymptote *does* intersect the graph of the function.

I'm willing to part with **5 bonus points** if you can find the point of intersection of $R(x)$ with its horizontal asymptote and label it with an ordered-pair label. I'm also looking for its effect on the graph. There's a little wiggle to this graph in the 1st quadrant.

1st: Domain:

$x^2+2x-15 \neq 0$

$(x+5)(x-3) \neq 0$

$x \neq -5$ and $x \neq 3$

$(-\infty,-5)\cup(-5,3)\cup(3,\infty)=\mathbb{R}\setminus\{-5,3\}$

Candidates for vertical Asymptotes

$x=-5$
$x=3$

But we have to check the numerator to see if any cancels. If it does, we get a hole, and not an asymptote.

Find zeros of $R(x)$:

$\dfrac{6x^2+x-77}{x^2+2x-15}$ $\underline{\underline{\text{SET}}}$ 0 $\Rightarrow$

$6x^2+x-77=0$

$-2\cdot3\cdot7\cdot11$
$=(2\cdot11)(3\cdot7)(-1)$

$2\cdot11-3\cdot7$
$22-21$

$6x^2+22x-21x-77$
$2x(3x+11)-7(3x+11)$
$=(3x+11)(2x-7)=0$

$\Rightarrow x = -\frac{11}{3}, + \frac{7}{2}$. Neither's a zero of the numerator.

No cancel.

No Hole.

$\therefore$ $\boxed{x = -5 \text{ \& } x = 3 \text{ are vertical asymptotes.}}$

$$\frac{6x^2 + x - 77}{x^2 + 2x - 15} = \frac{(3x+11)(2x-7)}{(x+5)(x-3)}$$

$\boxed{\begin{array}{l} x\text{-int: } \left(-\frac{11}{3}, 0\right), \left(\frac{7}{2}, 0\right) \\ y\text{-int: } \left(0, \frac{77}{15}\right) \end{array}}$

Horizontal Asymptote $\frac{6x^2}{x^2} \boxed{= 6 = y}$

Want to know what happens when $x \to \infty$ and $x \to -\infty$

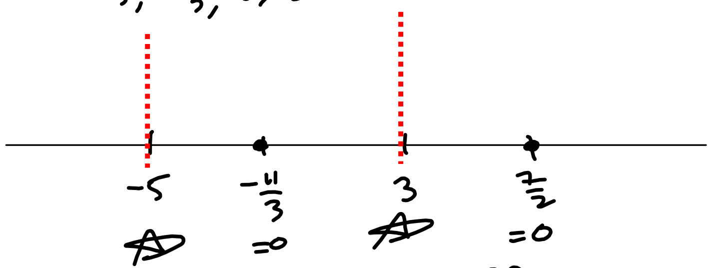
$$\frac{6x^2 + x - 77}{x^2 + 2x - 15} = \frac{x^2\left(6 + \frac{1}{x} - \frac{77}{x^2}\right)}{x^2\left(1 + \frac{2}{x} - \frac{15}{x^2}\right)} \xrightarrow{x \to \pm\infty} \frac{6 + 0 - 0}{1 + 0 - 0} = \frac{6}{1} = \boxed{\begin{array}{c} 6 = y \\ \text{Hor. As.} \\ \text{H.A.} \end{array}}$$

x-in: $(-\frac{11}{3},0)$, $(\frac{7}{2},0)$ "$=0$"

V.A.: $x=-5$, $x=3$ "$\nexists$"

Line 'em up!

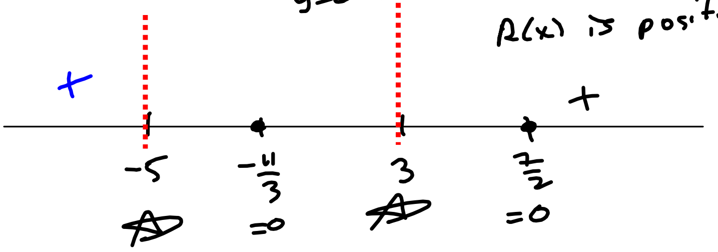
$-5, -\frac{11}{3}, 3, \frac{7}{2}$



Use H.A. to kick off your sign pattern.

$y=6$ means $R(x) \to 6$ for $x \to \pm\infty$

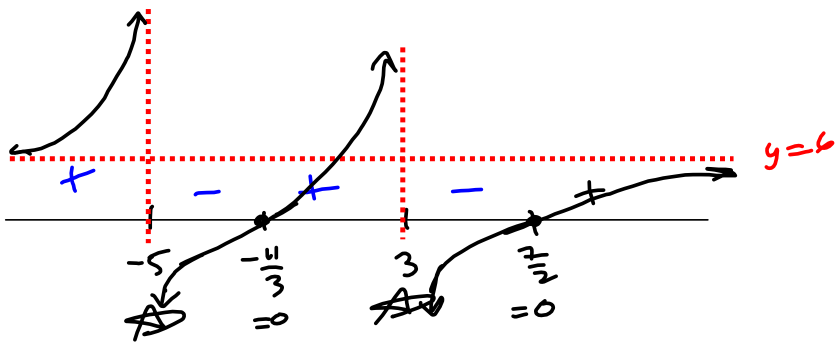
$R(x)$ is positive



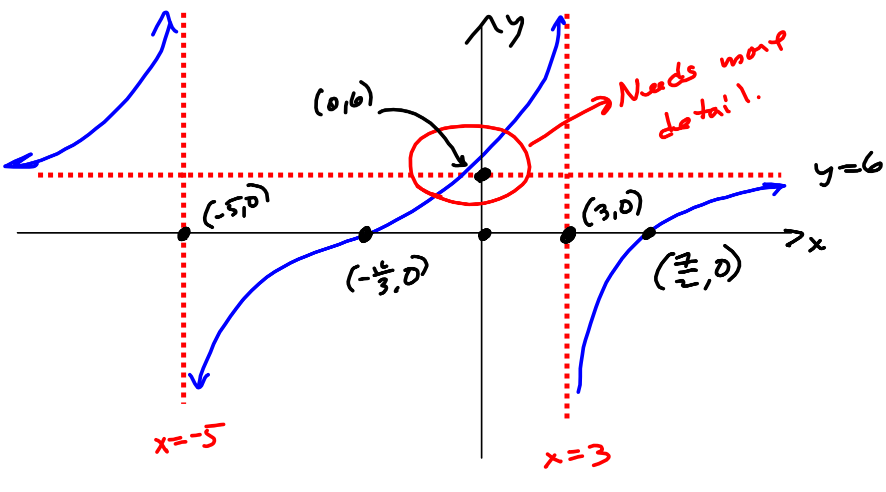
$$\frac{6x^2+x-77}{x^2+2x-15} = \frac{(3x+11)^1(2x-7)^1}{(x+5)^1(x-3)^1}$$

$(ax+b)^{\text{ODD}}$ changes sign @ $x=-\frac{b}{a}$

$(x-c)^{\text{EVEN}}$ doesn't cross @ $x=c$



Sketch it: Start With Asymptotes.



Bonus Opportunity for #8

I'm willing to part with **5 bonus points** if you can find the point of intersection of $R(x)$ with its horizontal asymptote and label it with an ordered-pair label. I'm also looking for its effect on the graph. There's a little wiggle to this graph in the 1st quadrant.

Find where $R(x)$ intersects $y=6$

Set $R(x) = \dfrac{6x^2+x-72}{x^2+2x-15} = 6 \Rightarrow$

$6x^2 + x - 72 = 6(x^2+2x-15) = 6x^2 + 12x - 90 \Rightarrow$

$-11x + 18 = 0 \Rightarrow$

$-11x = -18 \Rightarrow$

$\boxed{x = \frac{18}{11}} \Rightarrow$

$\left(\frac{18}{11}, 6\right)$

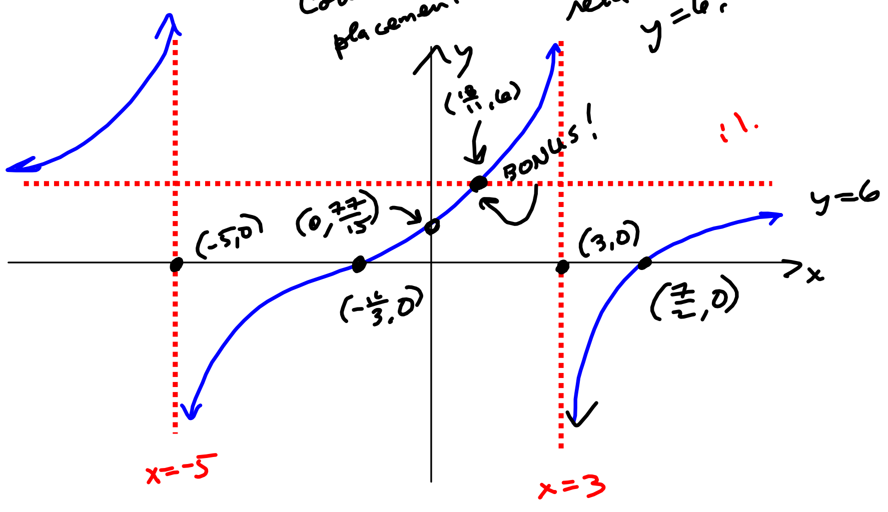
$$\frac{6\left(\frac{18}{11}\right)^2 + \frac{18}{11} - 77}{\left(\frac{18}{11}\right)^2 + 2\left(\frac{18}{11}\right) + 5}$$

$$= \frac{\frac{6(18^2)}{11^2} + \frac{18}{11}\cdot\frac{11}{11} - \frac{77}{1}\cdot\frac{11^2}{11^2}}{\frac{18^2}{11^2} + \frac{36}{11}\cdot\frac{11}{11} + \frac{5}{1}\cdot\frac{11^2}{11^2}}$$

This informs our graph

Looks painful

Corrected placement of y-axis relative to y=6.



9 (2 pts) Sketch the graph of $Q(x)=\dfrac{6x^3-11x^2-79x+154}{x^3-19x+30}$. Q has exactly the same graph as R, *except* for the *hole* in the graph of Q, which I expect you to find and clearly label in your graph. I'll give you full credit for #8 and #9, if you show the hole in the graph of Q on your sketch for R in #8 above.

$$\frac{6x^2+x-77}{x^2+2x-15} = \frac{(3x+11)(2x-7)}{(x+5)(x-3)} = R(x)$$

$Q(x) = R(x)$ with a hole

$$Q(x) = \left(\frac{(3x+11)(2x-7)}{(x+5)(x-3)}\right)\left(\frac{2x+b}{2x+b}\right)$$

$x^3-19x+30 = (x^2+2x+5)(x-c)$ for some c.

$= (x+5)(x-3)(x-c)$.

Use division to reveal $x-c$:

$x^3+0x^2-19x+30$

$x+5$ is a factor:

$$\begin{array}{r|rrrr} -5 & 1 & 0 & -19 & 30 \\ & & -5 & 25 & -30 \\ \hline 3 & 1 & -5 & 6 & 0 \\ & & 3 & -6 & \\ \hline & 1 & -2 & 0 & \end{array}$$

sweet!

$\to x-c = x-2$!

$c=2$!

x^2-5x+6
$=(x-2)(x-3)$
$c=2$!

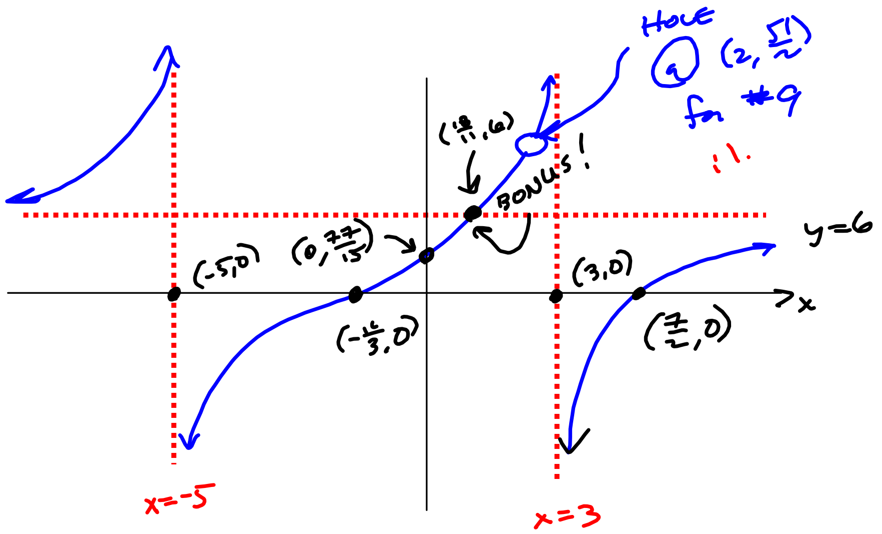
How do we find the y-coord of the hole?

Find $R(2)$!

$$\frac{6(2)^2+2-77}{2^2+2(2)-15} = \frac{24-75}{-7} = \frac{-51}{-7} = \frac{51}{7}$$

$\boxed{\text{Hole } \left(2, \frac{51}{2}\right)}$

$$\frac{6x^2+x-77}{x^2+2x-15} = \frac{(3x+11)(2x-7)}{(x+5)(x-3)}$$


HOLE @ (2, 51/2)
for #9
BONUS!
(0, 77/15)
(-5,0)
(3,0)
(7/2, 0)
y=6
x
x=-5
x=3

10. (5 pts) Sketch the graph of $T(x)=\dfrac{6x^3+13x^2-75x-154}{x^2+2x-15}$, showing all intercepts and asymptotes. This was also built off #8, so use the zeros you found for the numerator in #8 to help you find the 3rd zero of this new numerator.

#8

$$\frac{6x^2+x-77}{x^2+2x-15} = \frac{(3x+11)(2x-7)}{(x+5)(x-3)}$$

$$T(x) = \frac{(3x+11)(2x-7)(x-c)}{(x+5)(x-3)}$$

Don't know 'c' (or do we?)

Split off $3x+11$ & $2x-7$

$=3(x+\frac{11}{3})$ $\quad = 2(x-\frac{7}{2})$

$$\begin{array}{r|rrrr} -\frac{11}{3} & 6 & 13 & -75 & -154 \\ & & -22 & 33 & 154 \\ \hline \frac{7}{2} & 6 & -9 & -42 & 0 \\ & & 21 & 42 & \\ \hline & 6 & 12 & 0 & \end{array}$$

0 Sweet!

$(\cancel{+9}^{3})(+\frac{11}{\cancel{3}}) = 33$

$-\cancel{42}^{14}(-\frac{11}{\cancel{3}})$

$14 \cdot 14 = 154$

380, 38

This says

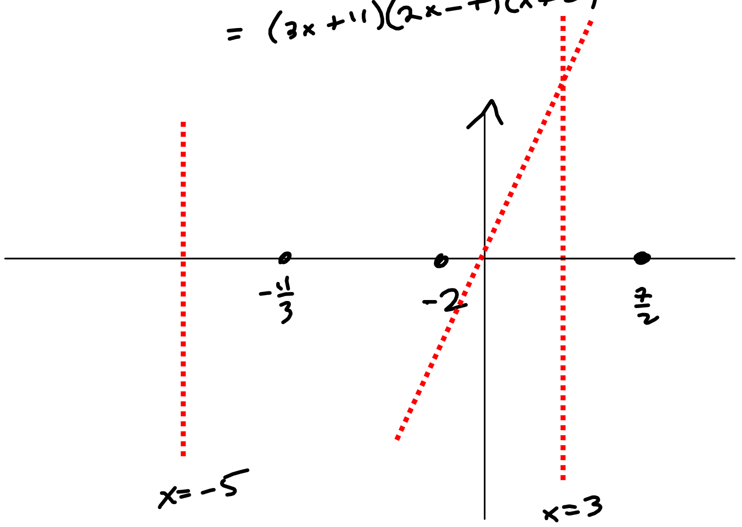
$$6x^3+13x^2-75x-154$$

$$= \left(x+\frac{11}{3}\right)\left(x-\frac{7}{2}\right)(6x+12)$$

$6 = 2 \cdot 3$

$$= 6\left(x+\frac{11}{3}\right)\left(x-\frac{7}{2}\right)(x+2)$$

$$= (3x+11)(2x-7)(x+2)$$



$$\frac{6x^3+13x^2-75x-154}{x^2+2x-15}$$ Divide.

$$\boxed{6x+1 = y \text{ is Slant Asymptote}}$$

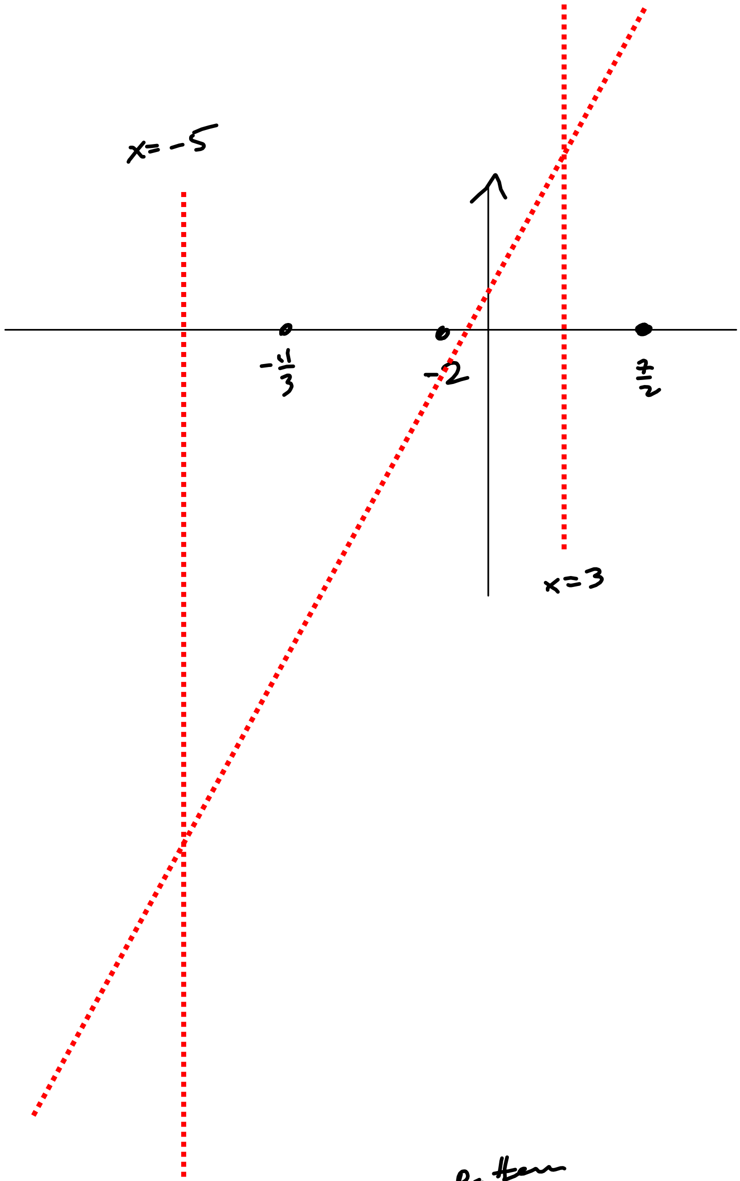
$$x^2+2x-15 \overline{\big)\, 6x^3+13x^2-75x-154}$$

$$-\left(6x^3+12x^2-90x\right)$$

$$x^2$$

$\frac{6x^3}{x^2} = 6x$

$\frac{x^2}{x^2} = 1$



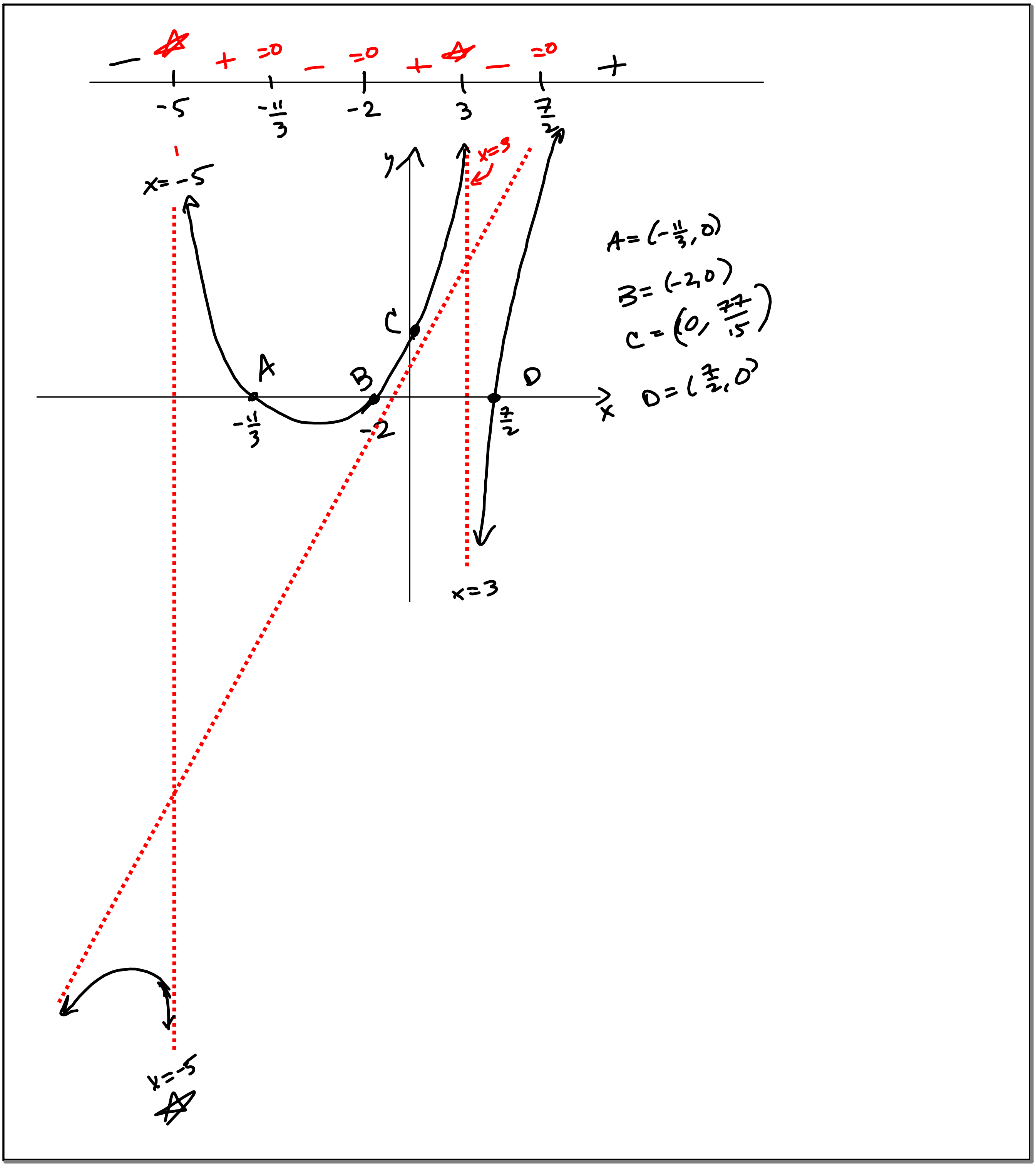
From Slant Asymptote — Sign Pattern

−	−5	$-\frac{11}{3}$	−2	3	$\frac{7}{2}$	+
	A	=0	=0	A	=0	

− (−5) + ($-\frac{11}{3}$) − (−2) + (3) − ($\frac{7}{2}$) +

$$\frac{(3x+11)^1(2x-7)^1(x+2)^1}{(x+5)^1(x-3)^1}$$

Alternating signs. All powers are odd



− + − + − +
=0 =0 =0
-5 -11/3 -2 3 7/2
x=-5
x=3
y
x
A
B
C
D
A=(-11/3, 0)
B=(-2,0)
C=(0, 77/15)
D=(7/2, 0)

11. (2 pts) What is the domain of $W(x)=\sqrt{(x+3)(x-3)^2(x+1)^3(x-8)^3}$?

$w(x)=\sqrt{(x+3)^1(x-3)^2(x+1)^3(x-8)^3} = \sqrt{G(x)}$

$G(x) \geq 0$

$G(x) = x^9 +$ lower degree

Need $(x+3)^1(x-3)^2(x+1)^3(x-8)^3 \geq 0$

$-3, -1, 3, 8$

N Y Y Y N Y N Y Y

$- \quad + \quad - \quad - \quad +$

$-3 \quad -1 \quad 3 \quad 8$

$=0 \quad =0 \quad =0 \quad =0$

≥ 0

$$\boxed{= [-3,-1] \cup \{3\} \cup [8,\infty) = \mathcal{D}(w)}$$

Need $\dfrac{(x+3)(x-8)^3}{(x-3)^2(x+1)^3}$

12. (2 pts) What is the domain of $K(x)=\sqrt{\dfrac{(x+3)(x-8)^3}{(x-3)^2(x+1)^3}}$?

N Y Y N N N N Y Y

$- \quad + \quad - \quad - \quad +$

$-3 \quad -1 \quad 3 \quad 8$

$=0 \quad ☆ \quad ☆ \quad =0$

≥ 0

$$\boxed{\mathcal{D}(K) = [-3,-1) \cup [8,\infty)}$$