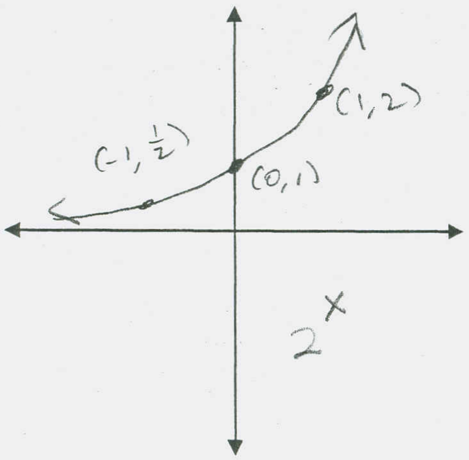
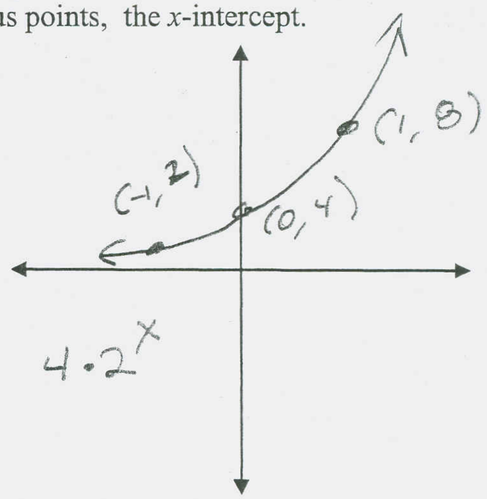
MAT 121-G81 | Test 4 – Spring, 2011 | Name________________________

100 Points | Covers Chapter 4, plus 7.1, 7.3 | NO GRAPHING CALCULATOR!!!

Graph:

1. (20 pts) Starting with $f(x) = 2^x$, sketch the graph of $g(x) = 4 \cdot 2^{x+1} - 5$ in 4 steps (counting $f(x) = 2^x$ as the first step). Use $x = -1$, $x = 0$, and $x = 1$ to find 3 points in the first graph, and show how these 3 points are moved around by each step in the transformation to $g(x)$. Your final graph should also show the y-intercept and, for 5 bonus points, the x-intercept.

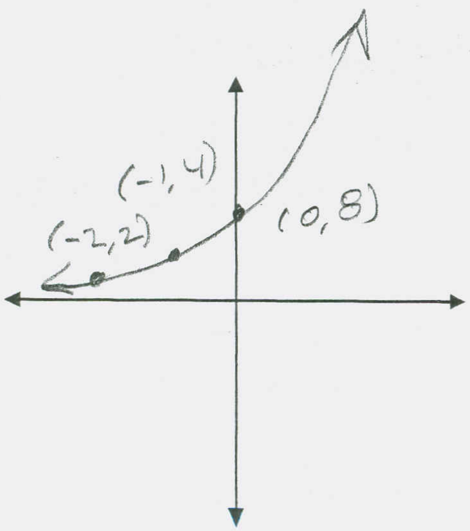


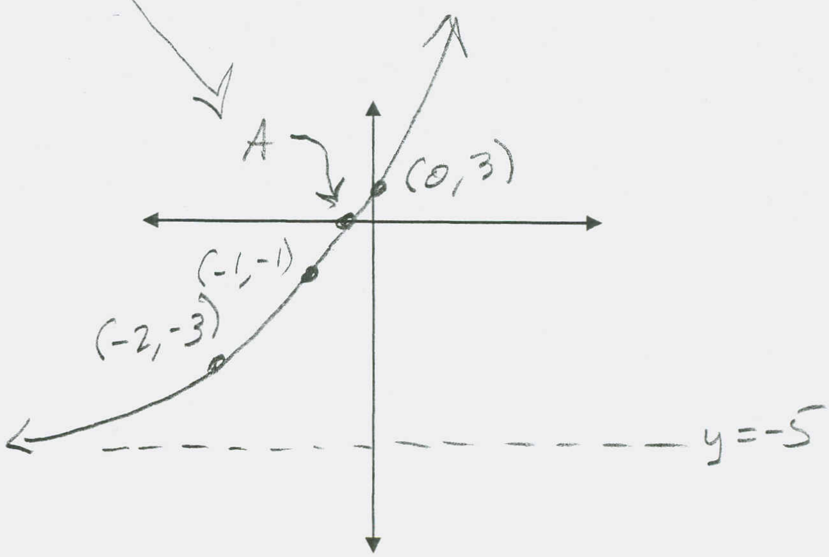


$$\boxed{\begin{aligned} A &= \left(\log_2\left(\tfrac{5}{4}\right) - 1,\ 0\right) \\ &\approx (-.678071905,\ 0) \end{aligned}}$$
5pts bonus

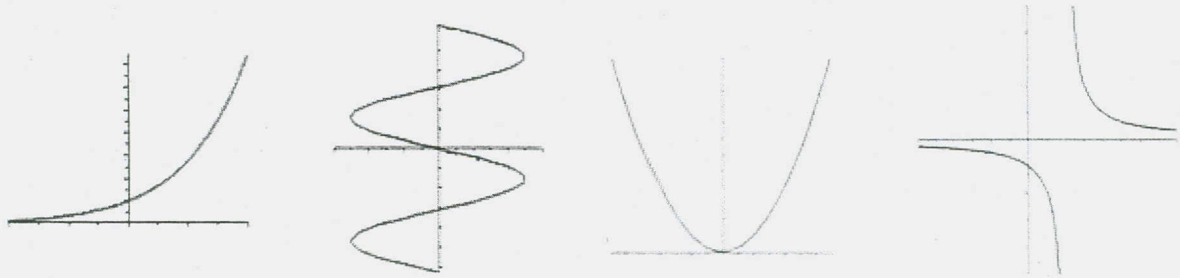
$$4 \cdot 2^{x+1} - 5 = 0$$
$$4 \cdot 2^{x+1} = 5$$
$$2^{x+1} = \frac{5}{4}$$

$$x + 1 = \log_2\left(\frac{5}{4}\right)$$
$$x = \log_2\left(\frac{5}{4}\right) - 1$$





2. (10 pts) Determine which of the following are functions and whether they are one-to-one. So indicate by writing "Yes" or "No" in the appropriate spaces.



Function? Yes	Function? No	Function? Yes	Function? Yes
1-to-1? Yes	1-to-1? No	1-to-1? No	1-to-1? Yes

3. For $f(x) = \sqrt{x-1}$ and $g(x) = 2x+1$, determine the following composite functions, *simplify them*, and state their domains:

a. (5 pts) $(f \circ g)(x) = \sqrt{(2x+1)-1} = \sqrt{2x}$

$\mathcal{D} = \{x \mid x \geq 0\} = [0, \infty)$

b. (5 pts) $(g \circ f)(x) = 2\sqrt{x-1} + 1$

$\mathcal{D} = \{x \mid x \geq 1\} = [1, \infty)$

4. (5 pts) What is the domain of $g(x) = \ln(-2x+3)$?

$$-2x+3 > 0$$
$$-2x > -3$$
$$x < \frac{3}{2}$$

$$\boxed{\mathcal{D} = \{x \mid x < \tfrac{3}{2}\} = (-\infty, -\tfrac{3}{2})}$$

5. (5 pts) What is the domain of $\ln\left(\dfrac{(x-2)^2}{(x-3)(x+1)^3}\right)$? (This is like a Chapter 3 question!)

$x=2$ $m=2$ touch
$x=3$ $m=1$ cross
$x=-1$ $m=3$ cross

$$x=4: \quad \frac{(4-2)^2}{(4-3)(4+1)^3} = \frac{2^2}{(1)(5)^3} = \text{positive} = +$$

$$\boxed{\mathcal{D} = (-\infty, -1) \cup (3, \infty)} = \boxed{\{x \mid x < -1 \text{ OR } x > 3\}}$$

6. (5 pts) Let $f(x) = 5^{x+1} - 4$. Find $f^{-1}(x)$.

$$5^{y+1} - 4 = x$$
$$5^{y+1} = x+4$$
$$y+1 = \log_5(x+4)$$
$$y = \boxed{\log_5(x+4) - 1 = f^{-1}(x)}$$

CHECK:

$$f(f^{-1}(x)) = 5^{(\log_5(x+4)-1)+1} - 4$$
$$= 5^{\log_5(x+4)} - 4 = x+4-4 = x \checkmark$$

7. (5 pts) Find functions *f* and *g* so that $f \circ g = H$, given that $H(x) = \ln(x^2 - 1)$.

$f(x) = \ln x$

$g(x) = x^2 - 1$

8. (5 pts) Evaluate $\log_2(96) - \log_2(3)$ *without a calculator* !!

$= \log_2\left(\frac{96}{3}\right) = \log_2(32) = \boxed{5,}$ b/c $32 = 2^5$.

9. (5 pts) Solve *without a calculator*: $5^{x-1} = 3^x$. All I want is a symbolic answer and the symbolic manipulations you perform to *get* there. For full credit, your answer should involve a logarithm or two in it.

$\log_5(5^{x-1}) = \log_5(3^x)$

$x - 1 = x \cdot \log_5(3)$

Let $a = \log_5(3)$. Then

$x - 1 = x \cdot a = ax \Rightarrow$

$x - ax = 1 \Rightarrow$

$x(1-a) = 1 \Rightarrow$

$x = \frac{1}{1-a} = \frac{1}{1-\log_5(3)} = \boxed{\frac{1}{1-\ln(3)/\ln(5)}}$

There's a $\log_3$ version. Here's the ln version:

$(x-1)\ln 5 = x \ln 3$

Let $a = \ln 5$, $b = \ln 3$. Then

$ax - a = bx \Rightarrow$

$ax - bx = a \Rightarrow$

$(a-b)x = a \Rightarrow x = \frac{a}{a-b} = \boxed{\frac{\ln 5}{\ln 5 - \ln 3}}$

same.

10. (5 pts) Write the following as the logarithm of a single expression. Assume that variables represent positive numbers. $3\log_5(x+7) - 2\log_5(x-7) + \log_5 9$

$= \log_5\left(\frac{9(x+7)^3}{(x-7)^2}\right)$

11. (10 pts) Solve: $\ln(x-4)+\ln(x+1)=\ln(6)$ for x.

$\ln((x-4)(x+1)) = \ln(6)$

$(x-4)(x+1) = 6$

$x^2 - 4x + x - 4 = 6$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

$x \in \{-2, 5\}$

$x=-2$ doesn't check.
$x=5$ does.

Final Answer:

$\boxed{x \in \{5\}}$

12. (10 pts) The half-life of carbon-14 is (approximately) 5500 years.

a. Derive the exponential decay model $A(t) = A_0 e^{-kt}$. The trick, here, is to find the decay rate, k, based on the half-life given.

$A(5500) = A_0 e^{-5500k} = \frac{1}{2} A_0$

$\boxed{A(t) = e^{-\frac{\ln(2)}{5500} t}}$

$e^{-5500k} = \frac{1}{2}$

$-5500k = \ln\left(\frac{1}{2}\right)$

$k = \frac{\ln(\frac{1}{2})}{-5500} = \boxed{\frac{\ln(2)}{5500} = k}$

$\approx 1.260267601 \times 10^{-4}$

$= .0001260267601$

b. Use your model from above to predict the age of an ancient fire pit, if a charcoal sample from the pit contains 20% of its original carbon-14. For ease of solving this problem, you may want to just use a symbolic k until the last step. Round your final answer to the nearest year.

$A(t) = A_0 e^{-kt} = .2 A_0$

$e^{-kt} = .2$

$-kt = \ln(.2)$

$t = \frac{\ln(.2)}{-k}$

$t = \frac{\ln(.2)}{-\left(\frac{\ln(2)}{5500}\right)}$

≈ 12770.60452

$\approx \boxed{12{,}771 \text{ yrs}}$

13. Find the geometric sums:

a. (5 pts) $2+\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{2187}$ (Be careful finding your a, r, and n in $a\cdot r^{n-1}$)

$$= 2\left(\tfrac{1}{3}\right)^0 + 2\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{3}\right)^2 + \cdots + 2\left(\tfrac{1}{3}\right)^7$$

$$= \sum_{k=1}^{8} 2\left(\tfrac{1}{3}\right)^{k-1} \qquad a = 2, \; r = \tfrac{1}{3}, \; n = 8$$

$$S = a\left(\frac{1-r^n}{1-r}\right) = 2\left(\frac{1-\left(\frac{1}{3}\right)^8}{1-\frac{1}{3}}\right)$$

$$= \frac{2\left(1-\left(\frac{1}{3}\right)^8\right)}{\frac{2}{3}} = 3\left(1-\left(\tfrac{1}{3}\right)^8\right) \approx \boxed{2.999542753}$$

$3\,|\,2187$
$3\,|\,729$
$3\,|\,243$
$3\,|\,81$
$3\,|\,27$
$3\,|\,9$
3

b. (5 pts) $\sum_{k=1}^{\infty} 7\cdot\left(\frac{3}{5}\right)^{k-1}$

$$S = a\left(\frac{1}{1-r}\right) = 7\left(\frac{1}{1-3/5}\right) = 7\left(\frac{1}{\frac{2}{5}}\right)$$

$$= 7\left(\frac{5}{2}\right) = \boxed{\frac{35}{2}} \text{ OR } 17.5$$