1340 WEEK 8 Written Solns MILLS

① We complete the square to write $f(x)$ in Standard Form ($a(x-h)^2+k$) and we sketch $f(x)$, showing all intercepts and its vertex (h,k). } content 1pt

ⓐ 5pts $f(x) = x^2-4x-21 = x^2-4x+2^2-4-21$

1pt

$= (x-2)^2-25 \Rightarrow \boxed{\text{vertex} = (h,k) = (2,-25)}$ and

$f(x)=0 \Rightarrow (x-2)^2 = 25$

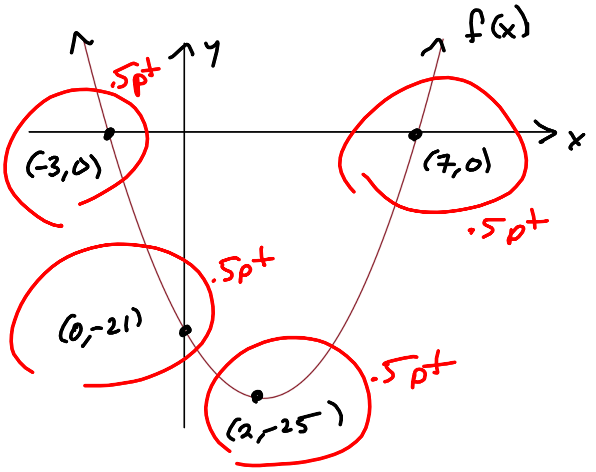
$\Rightarrow x-2 = \pm 25 = \pm 5$

$\Rightarrow x = 2 \pm 5$ → $x=7$ ⇝ (7,0) x-int

or

→ $x=-3$ ⇝ (−3,0) x-int

Also, $f(0) = -21 \longrightarrow \boxed{(0,-21) \text{ y-int}}$



.5pt label x- & y-axes

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(b) (5pts) $f(x) = -3x^2 - 42x - 72$

1pt If the wrote the instructions (Context) This means context counted 1pt for each of a & b

$= -3(x^2 + 14x \qquad) - 72$

$= -3(x^2 + 14x + 7^2) - (-3)(49) - 72$

Subtract this off

Scratch: $\begin{array}{r} {}^{2}49 \\ \times\ 3 \\ \hline 147 \end{array}$

1pt

$= -3(x+7)^2 + 75$

$147 - 72 = 75$

$\Longrightarrow (h,k) = \boxed{(-7,75) = \text{vertex}}$, and

$f(x) = 0 \Longrightarrow -3(x+7)^2 + 75 = 0$

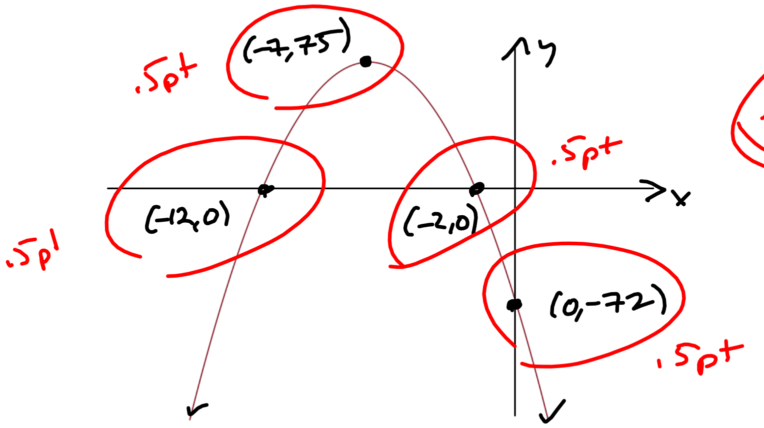
$\Longrightarrow -3(x+7)^2 = -75$

$\Longrightarrow (x+7)^2 = 25$

$\Longrightarrow x = -7 \pm 5$: $-2 = x$ OR $-12 = x$

$\boxed{(-2,0),\ (-12,0)\ \text{x-int's}}$

$f(0) = -72 \Longrightarrow \boxed{(0,-72)\ \text{is y-int}}$



.5 pt label x- & y-axes.

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(b) (5pts) Alternate Completing-the-square technique

$f(x) = -3x^2 - 42x - 72 = -3(x^2 + 14x + 24)$

$\longrightarrow \frac{f(x)}{-3} = x^2 + 14x + 24 = x^2 - 14x + 7^2 - 49 + 24$

$= (x-7)^2 - 25 = \frac{f(x)}{-3} \longrightarrow$

$\underline{f(x) = -3(x-7)^2 + 75}$, etc.

This presents another style for completing the square, preferred by some. Accepted by all.

(2) Let $f(x) = x^6 + 11x^5 + 19x^4 - 115x^3 - 200x^2 + 500x$ & $p(x) = x - 2$.

(a) (5pts) Find quotient $g(x)$ and remainder $r(x)$ for $f(x) \div p(x)$. Then write $f(x) = p(x)\,g(x) + r(x)$. USE SYNTHETIC DIVISION

Context (1pt)

Synthetic Division (2pts)

	6	5	4	3	2	1	0
2	1	11	19	−115	−200	500	0
		2	26	90	−50	−500	0
	1	13	45	−25	−250	0	$\boxed{0 = r(x)}$

This says $f(x) = (x-2)(x^5 + 13x^4 + 45x^3 - 25x^2 - 250x) + 0$, 2pts.

$r(x) = 0$ & $p(x) = x - 2$ is a factor of $f(x)$.

(b) (5pts) Crap. I wanted a $p(x)$ that <u>wasn't</u> a factor.

In any case, $R(x) = \frac{f(x)}{p(x)} = x^5 + 13x^4 + 45x^3 - 25x^2 - 250x = g(x)$

and there's no $\frac{r(x)}{p(x)}$ at all. So $\frac{f(x)}{p(x)} = g(x)$ is a polynomial everywhere. If they get confused b/c $r(x)=0$, don't hurt them.

REMARKS FOR STUDENT ENRICHMENT:

I was aiming for something more like $p(x) = x - 3$

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We use synthetic division to divide f by x - 3:

$$\begin{array}{r|rrrrrrr} 3 & 1 & 11 & 19 & -115 & -200 & 500 & 0 \\ & & 3 & 42 & 183 & 204 & 12 & 1536 \\ \hline & 1 & 14 & 61 & 68 & 4 & 512 & \boxed{1536 \text{ - remainder}} \end{array}$$

This says $f(x) = (x-3)(x^5 + 14x^4 + 61x^3 + 68x^2 + 4x + 512) + 1536$

This is why the Remainder Theorem, which says

$$f(3) = (3-3)(3^5 + 4(3)^4 + 61(3)^3 + 68(3)^2 + 4(3) + 512) + 1536$$

$= \boxed{f(3) = 1536}$ = remainder, when $f(x)$ is divided by $x-3$.

Furthermore $\dfrac{f(x)}{p(x)} = x^5 + 14x^4 + 61x^3 + 68x^2 + 4x + 512 + \dfrac{1536}{x-3}$

So this rational function is basically a polynomial plus a $\dfrac{1536}{x-3}$ term that vanishes as $x \to \pm\infty$.

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c (5pts) Based on 2a, $\boxed{f(2)=0}$ Easiest one!

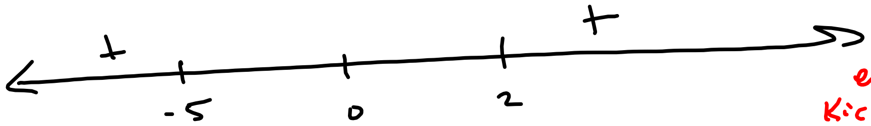
d (5pts) End behavior of f is controlled by the leading term x^6, whose end behavior is

Also, easiest one.

Context 1pt

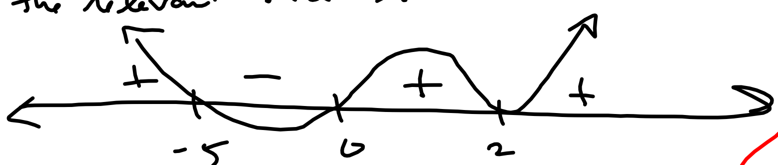
e (5pts) $f(x)=(x)(x-2)^2(x+5)^3$ is factored form of f. We sketch $f(x)$ using its x- & y-intercepts, y-intercept, and end behavior

x^6 gives this much of the sign pattern



This is to show students how end behavior kick-starts the sign pattern

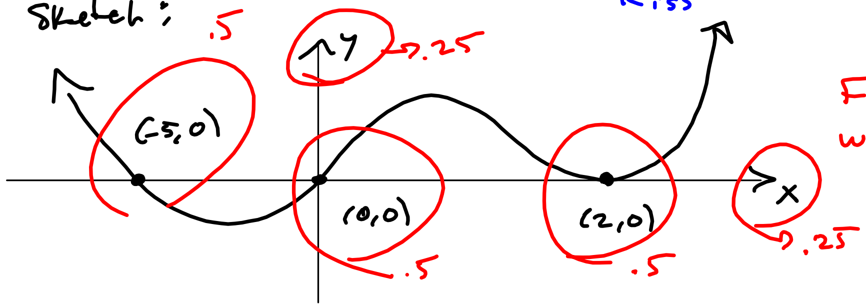
You get the rest by looking for odd/even powers of the relevant factors.



2pts Need to see the intervals and the +'s & −'s in the right place

$(x+5)^3$	x^1	$(x-2)^2$
3 is odd	1 is odd	2 is even
sign changes	sign changes	sign doesn't change.
cross	cross	kiss

Sketch:



Finished graph with all labels

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2 f 5pts $D(x) = x^2 - 3x - 15$. We find the quotient $Q(x)$ and remainder $R(x)$, when $f(x)$ is divided by $D(x)$.

1pt context.

2pts QUOTIENT $Q(x)$

$$\begin{array}{r}
\boxed{x^4 + 14x^3 + 76x^2 + 323x + 1909} \\
x^2 - 3x - 15 \,\big)\, x^6 + 11x^5 + 19x^4 - 115x^3 - 200x^2 + 500x \\
-(x^6 - 3x^5 - 15x^4) \\
\hline
14x^5 + 34x^4 - 115x^3 - 200x^2 + 500x \\
-(14x^5 - 42x^4 - 210x^3) \\
\hline
76x^4 + 95x^3 - 200x^2 + 500x \\
-(76x^4 - 228x^3 - 1140x^2) \\
\hline
323x^3 + 940x^2 + 500x \\
-(323x^3 - 969x^2 - 4845x) \\
\hline
1909x^2 + 5345x \\
-(1909x^2 - 5727x - 28635) \\
\hline
11072x + 28635
\end{array}$$

REMAINDER $\boxed{R(x) = 11072x + 28635}$ 2pts

$76 \times 3 = 228$

$(76)(15) = 760 + 380 = 1140$

$(15)(323) = 3230 + 1615 = 4845$

$\frac{3230}{2} = 1615$, $1615 + 3230 = 4845$

$1909 \times 3 = 5727$

$(15)(1909) = 19090 + 9545 = 28635$

$\frac{19090}{2} = 9545$

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The above work shows that

$$f(x) = D(x)\,Q(x) + R(x)$$

$$= (x^2-3x-15)(x^4+14x^3+76x^2+323x+1909) + 11072x + 20635$$

3.6 Preview !

$$\frac{f(x)}{D(x)} = \frac{x^6+11x^5+19x^4-115x^3-200x^2+500x}{x^2-3x-15}$$

$$= x^4+14x^3+76x^2+323x+1909 + \frac{11072x+20635}{x^2-3x-15}$$

Now, as $x \longrightarrow \pm\infty$, $\frac{f(x)}{D(x)} \longrightarrow x^4+14x^3+76x^2+323x+1909$

because $\frac{11072x+20635}{x^2-3x-15} \longrightarrow 0$ as $x \longrightarrow \pm\infty$

(x^2 out-grows $11072x$)

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g 5pts $f(x) = x(x-2)^2(x+5)^3$ → Context 2pts

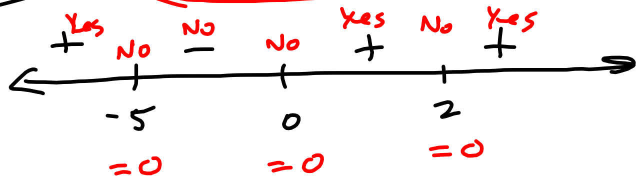
$x=0$ is a zero of multiplicity $m=1$,
$x=2$ " " " " " $m=2$, and
$x=-5$ " " " " " $m=3$.

3pts

h Bonus 5pts

Context 1pt

Solve $f(x) > 0$



>0

Sign pattern & the yes's & no's. 3pts

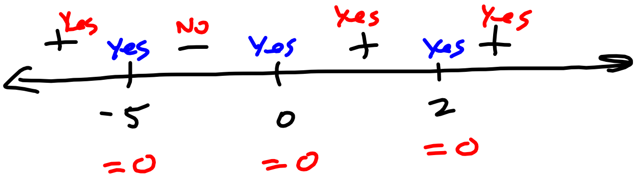
→ $x \in (-\infty, -5) \cup (0,2) \cup (2, \infty)$

Final Answer: 1pt

i Bonus 5pts We find the domain $\mathcal{D}(g)$ for $g(x) = \sqrt{f(x)}$

Context 1pt

→ We need $f(x) \geq 0$ 1pt



≥ 0

2pts sign pattern & yes's & no's (supporting analysis)

→ $\boxed{\mathcal{D}(g) = (-\infty, -5] \cup [0, \infty)}$

Final Ans. 1pt