$$f(x) = 4x^5 + 13x^4 - 56x^3 + 281x^2 - 410x + 168$$

1. (2 pts) Describe the end behavior of f with a simple graphic.

$4x^5$

2. (2 pts) Use Descartes' Rule of Signs to determine the *possible* number of positive and negative zeros.

$$f(x) = 4x^5 \underbrace{+13x^4 - }_{1}\underbrace{56x^3 +}_{2}\underbrace{281x^2 - }_{3}\underbrace{410x + }_{4}168$$

$\boxed{\text{4, 2 or 0 positive zeros}}$

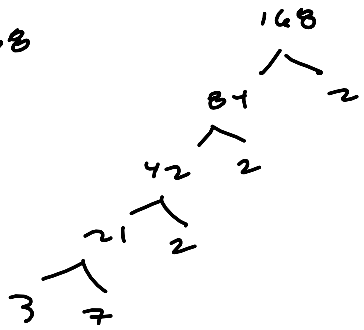
$$f(-x) = \underbrace{-4x^5 +}_{1} 13x^4 + 56x^3 + 281x^2 + 410x + 168$$

$\boxed{\text{Exactly 1 negative zero}}$

3. (2 pts) Use the Rational Zeros Theorem to determine the *possible* rational zeros (roots) of f.

$$f(x) = 4x^5 + 13x^4 - 56x^3 + 281x^2 - 410x + 168$$

$a_n = 4,\ a_0 = 168$



$168 = 3 \cdot 7 \cdot 2 \cdot 2 \cdot 2$

ugh!

If $\frac{p}{q}$ is a zero of f, then

p is a divisor of $a_0 = 168$ &

q is a divisor of $a_n = 4$

$168 = 3 \cdot 7 \cdot 2 \cdot 2 \cdot 2$

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2, \pm \cancel{\frac{2}{2}}, \pm \cancel{\frac{2}{4}}, \pm 4, \pm \cancel{\frac{4}{2}}, \pm \cancel{\frac{4}{4}}, \pm 8, \pm \cancel{\frac{8}{2}}, \pm \cancel{\frac{8}{4}},$

$\pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm 6, \pm \cancel{\frac{6}{2}}, \pm \cancel{\frac{6}{4}}, \pm 12, \pm \cancel{\frac{12}{2}}, \pm \cancel{\frac{12}{4}}, \pm 24, \pm \cancel{\frac{24}{2}}, \pm \cancel{\frac{24}{4}},$

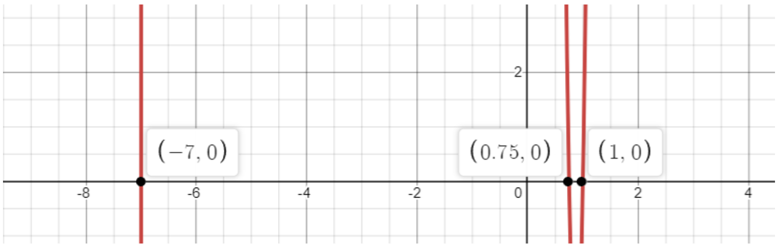
$\pm 7, \pm \frac{7}{2}, \pm \frac{7}{4}, \pm 14, \pm \cancel{\frac{14}{2}}, \pm \cancel{\frac{14}{4}}, \pm 28, \pm \cancel{\frac{28}{2}}, \pm \cancel{\frac{28}{4}}, \pm 56, \pm \cancel{\frac{56}{2}}, \pm \cancel{\frac{56}{4}},$

$\pm 21, \pm \frac{21}{2}, \pm \frac{21}{4}, \pm 42, \pm \cancel{\frac{42}{2}}, \pm \cancel{\frac{42}{4}}, \pm 84, \pm \cancel{\frac{84}{2}}, \pm \cancel{\frac{84}{4}}, \pm 168, \pm \cancel{\frac{168}{2}}, \pm \cancel{\frac{168}{4}}$

The "cheat," here, is to narrow down your guesses with some kind of graphing utility. This will allow us to combine technology and our knowledge of rational zeros to break *f* down all the way (Split it into linear factors).

4. (2 pts) Using the information, above, find all real zeros of f. Finding all zeros includes finding the multiplicity of each. This means performing multiple synthetic divisions. Always check for multiplicity greater than 1 with another synthetic division, just in case.

$$f(x) = 4x^5 + 13x^4 - 56x^3 + 281x^2 - 410x + 168$$



Desmos is telling us x = -7, 0.75, and 1 are zeros. For our work, we will convert the 0.75 to 3/4. Now, we divide by (x + 7), (x - 1), and then (x -3/4).

$$\begin{array}{r|rrrrrr}
-7 & 4 & 13 & -56 & 281 & -410 & 168 \\
 & & -28 & 105 & -343 & 434 & -168 \\ \hline
1 & 4 & -15 & 49 & -62 & 24 & 0 \\
 & & 4 & -11 & 38 & -24 & \\ \hline
\frac{3}{4} & 4 & -11 & 38 & -24 & 0 & \\
 & & 3 & -6 & 24 & & \\ \hline
 & 4 & -8 & 32 & 0 & &
\end{array}$$

sweet! sweet! sweet!

Quadratic: $4x^2 - 8x + 32 \overset{\text{SET}}{=} 0$

$\Rightarrow 4(x^2 - 2x + 8) = 0 \rightarrow$

$a = 1, b = -2, c = 8 \rightarrow$

$b^2 - 4ac = (-2)^2 - 4(1)(8) = 4 - 32 = -28 < 0$

Irreducible over reals.

No more real zeros, so,

$\boxed{x = -7, \frac{3}{4}, 1 \text{ are the real zeros.}}$

5. (2 pts) From your work, above, factor f over the real numbers. This will involve an irreducible quadratic factor that your grapher has no way of helping you to see. without the synthetic divisions in #4, bringing you closer and closer, step by step, to the irreducible quadratic.

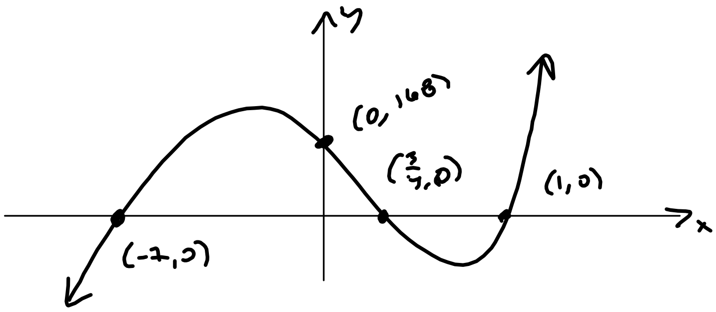
This work says

$\boxed{f(x) = (x+7)(x-\frac{3}{4})(x-1)(4x^2 - 8x + 32)}$

OR $\boxed{4(x+7)(x-\frac{3}{4})(x-1)(x^2 - 2x + 8)}$

Either one.

6. (2 pts) Give a rough sketch of f from all of the above information. This is an *art* whose essence is really only found in my videos. If you're too tied to your grapher's output, you'll not capture the real essence of what's going on, or the key features I'm always looking for. Your picture will be more "vertical" than it should be.



This isn't *exactly* what it looks like, but it captures the shape and the *x*- and *y*-intercepts. By the time you made it tall enough to include the *actual* maximum, the scale would be too large to see $x = 1$ and $x = 3/4$ as separate points, let alone any part of the graph that's below the *x*-axis, even though we *know* there's a tiny bit there under the *x*- axis between $x = 3/4$ and $x = 1$.

7. (2 pts)

Now do your quadratic equation thing to find the 2 nonreal roots. Finally, apply the Factor Theorem to *all* the above work, and represent f as a product of linear factors, $f(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_w)^{m_w}$.

$4x^2 - 8x + 32 = 0 \Rightarrow$

$4(x^2 - 2x + 4) = 0 \Rightarrow$

$x^2 - 2x + 4 = 0 \Rightarrow$

$x^2 - 2x + 1^2 - 1 + 4 = (x-1)^2 + 3 = 0 \Rightarrow$

$(x-1)^2 = -3 \Rightarrow$

$x - 1 = \pm\sqrt{-3} = \pm i\sqrt{3}$

$\Rightarrow \boxed{x = 1 \pm i\sqrt{3} \text{ or } 1 \pm \sqrt{3}\, i}$

$\Rightarrow \boxed{f(x) = 4(x+7)\left(x - \frac{3}{4}\right)(x-1)(x - (1+\sqrt{3}\, i))(x - (1 - \sqrt{3}\, i))}$

8. (5 pts) Sketch the graph of $R(x)=\dfrac{10x^2-59x+84}{3x^2-7x-10}$, showing all intercepts, asymptotes, and capturing the *essential features* of the shape of the graph. If you're a slave to your grapher, and oblivious to the features I'm looking for, it'll jump off the page at me (and be bad).

Get the factoring out of the way:

$10x^2-59x+84 =$

Scratch! $2\cdot5\cdot2\cdot2\cdot3\cdot7$

$$2\,\underline{|84}=\quad 2\,\underline{|42}\quad 3\,\underline{|21}\quad 7$$

Hit $59 =$

Sum	Product (want 840)
$58+1$	58
$49+10$	490
$39+20$	780
$\sim 35+24$	840 !

$24 \cdot 35$: 140, 700, 840

$3x^2-7x-10$
$=(3x-10)(x+1)$

$10x^2-35x-24x+84$
$=5x(2x-7)-12(2x-7)$
$=\boxed{(2x-7)(5x-12)}$

So, $R(x)=\dfrac{(2x-7)(5x-12)}{(3x-10)(x+1)}$

Domain:

$(3x+10)(x+1)\neq 0$
$3x+10\neq 0$ & $x+1\neq 0$
$3x\neq -10$ & $x\neq -1$
$x\neq -\frac{10}{3}$ & $x\neq -1$

$$\boxed{\begin{aligned}\mathcal{D}&=(-\infty,-1)\cup(-1,\tfrac{10}{3})\cup(\tfrac{10}{3},\infty)\\ &=\mathbb{R}\setminus\{\tfrac{10}{3},-1\}\\ &=\{x \mid x\neq \tfrac{10}{3} \text{ and } x\neq -1\}\end{aligned}}$$ Any of these.

None of the factors cancel, so $\boxed{x=\frac{10}{3} \text{ \& } x=-1}$ are the equations of the $\boxed{\text{vertical asymptotes.}}$

Zeros:

$R(x)=0 \Longrightarrow \dfrac{(2x-7)(5x-12)}{(3x-10)(x+1)}=0$

$\Longrightarrow (2x-7)(5x-12)=0$
$\Longrightarrow 2x-7=0$ or $5x-12=0$
$\Longrightarrow 2x=7$ or $5x=12$
$\Longrightarrow x=\frac{7}{2}$ or $x=\frac{12}{5}$

Zeros $x=\frac{7}{2},\frac{12}{5} \longrightarrow$ $\boxed{\text{x-ints are } (\tfrac{7}{2},0),\ (\tfrac{12}{5},0)}$

$\boxed{\text{y-int is } (0,-\tfrac{42}{5})}$ $\frac{84}{-10}=-\frac{42}{5}$ or $(0,-8.4)$

Finally, end behavior. Any horizontal or slant asymptotes?

What happens when *x* is very far to the left or very far to the right of the *y*-axis?

The degree of the numerator and denominator are the same. Easiest is to just divide the leading coefficients, as if the lower-degree terms don't exist.

$\dfrac{10x^2+\text{smaller}}{3x^2+\text{smaller}} \overset{|x|\text{ big}}{\approx} \dfrac{10x^2}{3x^2}=\dfrac{10}{3}$ degree = 2 up- & down-stairs

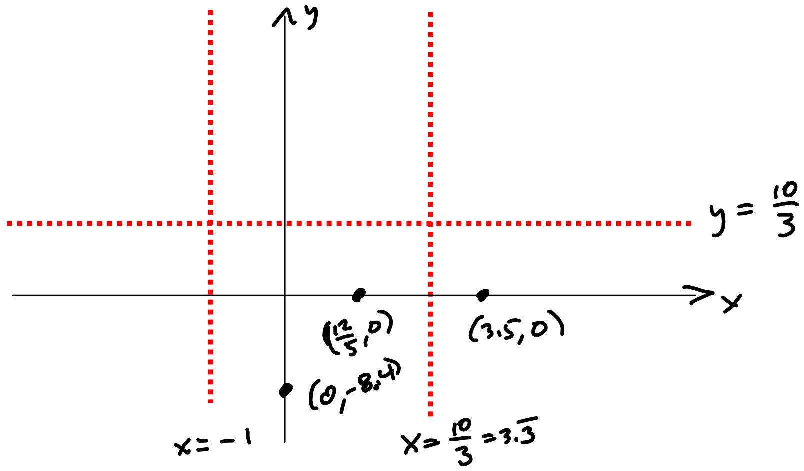
There are other ways of doing this, for instance, dividing through numerator and denominator by x^2.

$$\lim_{|x|\to\infty}\frac{x^2\left(10-\frac{59}{x}+\frac{84}{x^2}\right)}{x^2\left(3-\frac{7}{x}-\frac{10}{x^2}\right)}=\lim_{|x|\to\infty}\frac{10-\frac{59}{x}+\frac{84}{x^2}}{3-\frac{7}{x}-\frac{10}{x^2}}=\frac{10-0+0}{3-0-0}=\frac{10}{3}$$

FACT $\displaystyle\lim_{|x|\to\infty}\frac{a}{x^n}=0$ for any number a & any number $n>0$.

Horizontal Asymptote: $\boxed{y=\frac{10}{3}}$

Combine this work:



$\frac{12}{5} = 2.4$

$\frac{7}{2} = 3.5$

This is plenty to graph the whole thing, since we know all the factors associated with zeros and asymptotes are to the first power, because 1 is an odd number, so we know the *sign* of the graph changes at each of the cut points:

Zeros: $\frac{12}{5} = 2.4$, $\frac{7}{2} = 3.5$

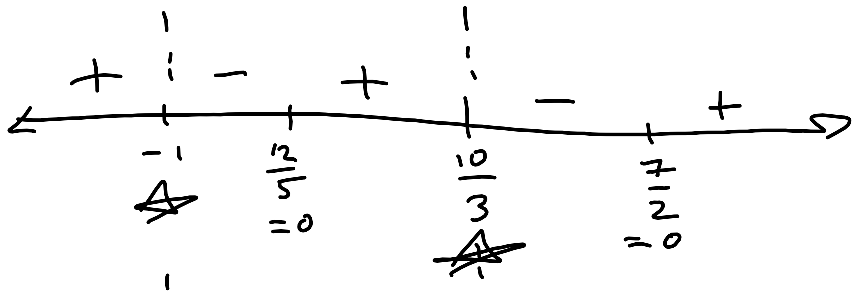
Asymptotes: $\frac{10}{3} = 3.\overline{3}$, -1

Arrange, left to right:

$-1, 2.4, 3.\overline{3}, 3.5$

$-1, \frac{12}{5}, \frac{10}{3}, \frac{7}{2}$

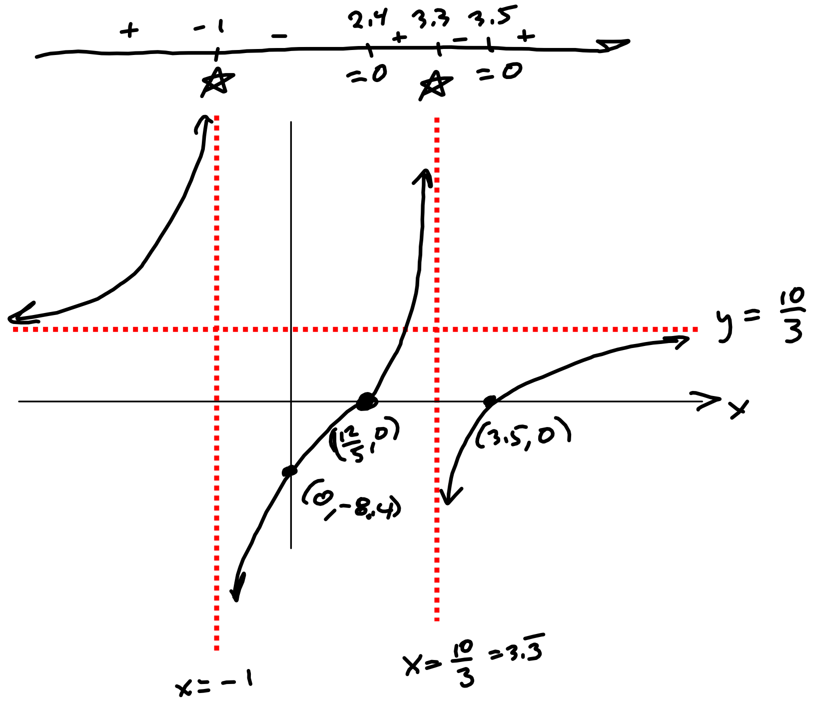
A $\quad =0 \quad\quad =0$



You know it's positive far right and far left by the horizontal asymptote.

Also, the multiplicity of all the factors/zeros is 1, so sign changes across all of the cut points.

Finally, the y-intercept of - 8.4 or - 42/5, which confirms the negative sign between *x* = -1 and *x* = 12/5. This is enough to obtain a rough sketch, by combining with the intercepts and asymptotes we created.


+
-1
-
2.4
+
3.3
-
3.5
+
=0
=0
y = 10/3
x
(12/5, 0)
(3.5, 0)
(0, -8.4)
x = -1
x = 10/3 = 3.3̄

I'm willing to part with **5 bonus points** if you can find the point of intersection of $R(x)$ with its horizontal asymptote and label it with an ordered-pair label. I'm also looking for its effect on the graph. There's a little wiggle to this graph in the 1[st] quadrant.

This didn't have the wiggle I was expecting, off to the right, but we have at least one crossing of the horizontal asymptote by the graph.

$$R(x) = \frac{10x^2-59x+84}{3x^2-7x-10} = \frac{10}{3}$$

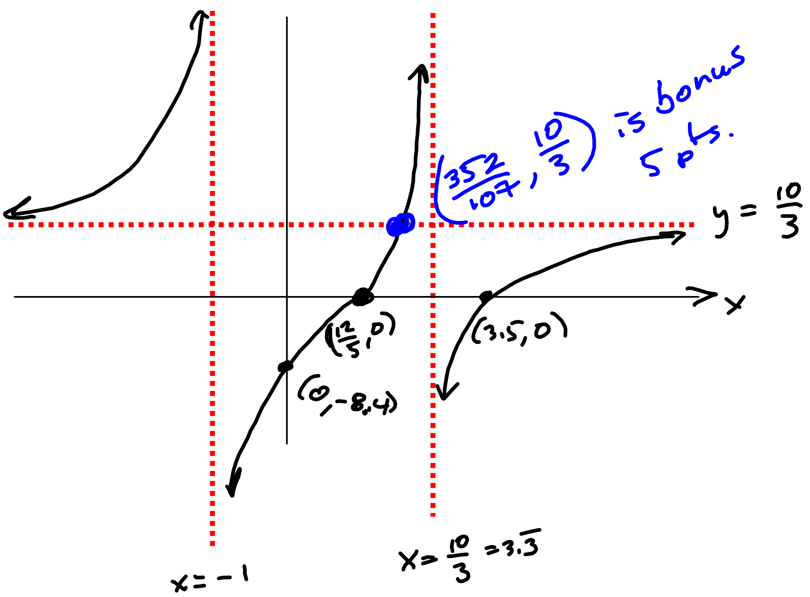
$$3(10x^2-59x+84) = 10(3x^2-7x-10)$$

$$30x^2-177x+252 = 30x^2-70x-100$$

$$-30x^2+70x-252 = -30x^2+70x-252$$

$$-107x = -352$$

$$x = \frac{352}{107}$$ $\left(\frac{352}{107}, \frac{10}{3}\right)$ is where $R(x) = \frac{10}{3}$



9. (2 pts) Sketch the graph of $Q(x)=\dfrac{10x^3-159x^2+674x-840}{3x^3-37x^2+60x+100}$. Q has exactly the same graph as R, *except* for the *hole* in the graph of Q, which I expect you to find and clearly label in your graph. I'll give you full credit for #8 and #9, if you show the hole in the graph of Q on your sketch for R in #8 above.

this means $Q(x)=\dfrac{(2x-7)(5x-12)(x-c)}{(3x-10)(x+1)(x-c)}$

To reveal the x - c, we divide the denominator by $x + 1$. That'll reduce finding c to solving a quadratic equation:

$$\begin{array}{r|rrrr} -1 & 3 & -37 & 60 & 100 \\ & & -3 & 40 & -100 \\ \hline & 3 & -40 & 100 & 0 \end{array}$$

so, $3x^3-37x^2+60x+100=(x+1)(3x^2-40x+100)$

$a=3, b=-40, c=100$

$b^2-4ac=40^2-4(3)(100)=1600-1200=400=20^2 \longrightarrow$

$$x=\frac{40\pm20}{2(3)}=\frac{20\pm10}{3} \begin{cases} \frac{30}{3}=10=c! \\ \frac{10}{3} \end{cases}$$

Check numerator for a factor of $x-10$, to be sure:

$$\begin{array}{r|rrrr} 10 & 10 & -159 & 674 & -840 \\ & & 100 & -590 & 840 \\ \hline & 10 & -59 & 84 & 0 \end{array}$$ sweet!

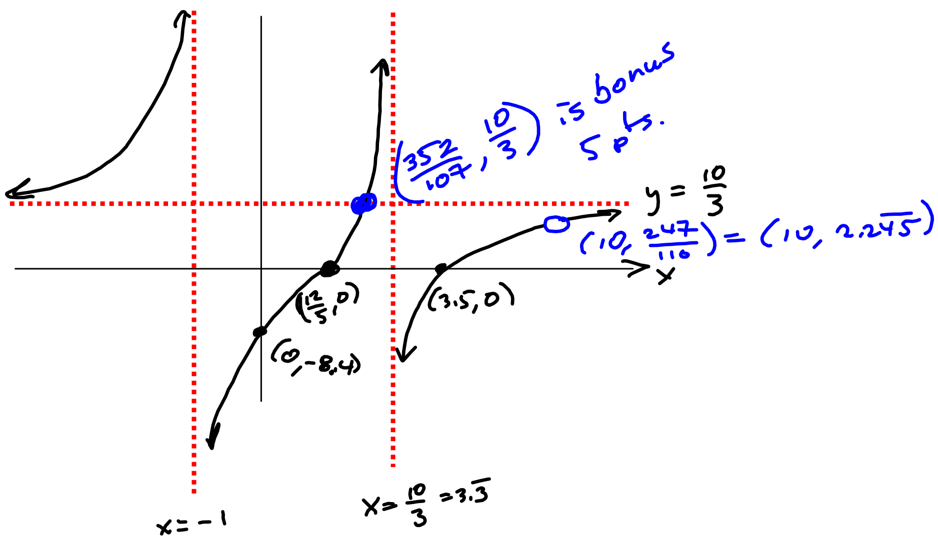
So, we just have to plug $x=10$ into $R(x)$ (essentially)

$$Q(x)=\frac{(2x-7)(5x-12)(x-10)}{(3x-10)(x+1)(x-10)}=\frac{(2x-7)(5x-12)}{(3x-10)(x+1)}=R(x), \text{ when}$$

$x\neq10$, so that's where the hole is, $R(10)=$

$$=\frac{(2(10)-7)(5(10)-12)}{(3(10)-10)(10+1)}=\frac{(13)(38)}{(20)(11)}=\frac{(13)(19)}{(10)(11)}=\frac{247}{110}\approx 2.245454545$$

HOLE: $\left(10,\frac{247}{110}\right)=(10,2.2\overline{45})$



10. (5 pts) Sketch the graph of $T(x)=\dfrac{10x^3-139x^2+556x-672}{3x^2-7x-10}$, showing all intercepts and asymptotes. This was also built off #8, so use the zeros you found for the numerator in #8 to help you find the 3[rd] zero of this new numerator.

$x = \frac{12}{5}$ & $\frac{7}{2}$

$$\begin{array}{r|rrrr} \frac{12}{5} & 10 & -139 & 556 & -672 \\ & & 24 & -276 & 672 \\ \hline \frac{7}{2} & 10 & -115 & 280 & 0 \\ & & 35 & -280 & \\ \hline & 10 & -80 & 0 & \end{array}$$

$\frac{115}{5} = 23 \cdot 12$ $\quad \frac{280}{5} = 56 \cdot 12$ $\quad 560 + 112 = 672$

So $10x^2 - 139x^2 + 556x - 672 = (10x-80)(x-\frac{12}{5})(x-\frac{7}{2})$ set $=0$ ⇒

$\boxed{x = \frac{12}{5}, \frac{7}{2}, 8}$ are zeros.

All multiplicity $m=1$, so sign changes.

$= 10(x-8)(x-\frac{12}{5})(x-\frac{7}{2})$

$= (x-8)(5x-12)(2x-7)$

Denominator: $(3x-10)(x+1)$

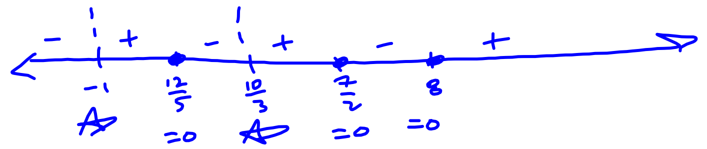
V.A.: $\boxed{x = \frac{10}{3}, -1}$

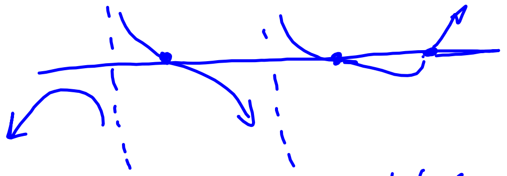
Cut Points:

$-1, \frac{10}{3}, \frac{12}{5}, \frac{7}{2}, 8$ Re-arrange left-to-right:

A A $=0$ $=0$ $=0$

$-1, \frac{12}{5}, \frac{10}{3}, \frac{7}{2}, 8$





Now for slant asymptote:

$$\begin{array}{r|l} & \frac{10}{3}x - \frac{347}{9} \\ \hline 3x^2-7x-10 & 10x^3 - 139x^2 + 556x - 672 \\ & -(10x^3 - \frac{70}{3}x^2 - \frac{100}{3}x) \\ \hline & -\frac{347}{3}x^2 \end{array}$$

$\frac{10x^3}{3x^2}$ $\qquad \frac{-\frac{347}{3}x^2}{3x^2} = -\frac{347}{9}$

$-139 + \frac{70}{3} = \frac{-139(3)+70}{3} = \frac{-417+70}{3} = \frac{-347}{3}$

Slant asymptote: $y = \frac{10}{3}x - \frac{347}{9}$

General idea of what it looks like:

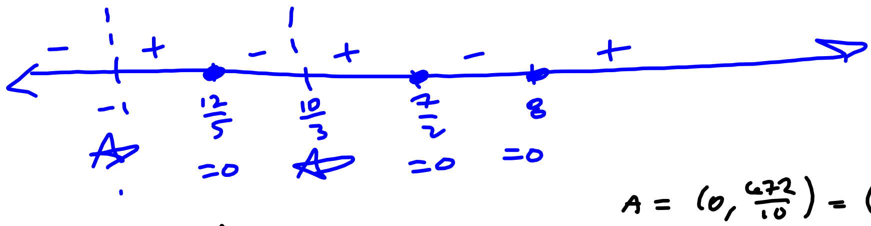
y-int $(0, -\frac{347}{9}) \approx (0, -38.55555556)$

x-int $\frac{10}{3}x - \frac{347}{9} = 0$

$30x - 347 = 0$

$30x = 347$

$x = \frac{347}{30} > \frac{300}{30} = 10$, so it's well off to the right.

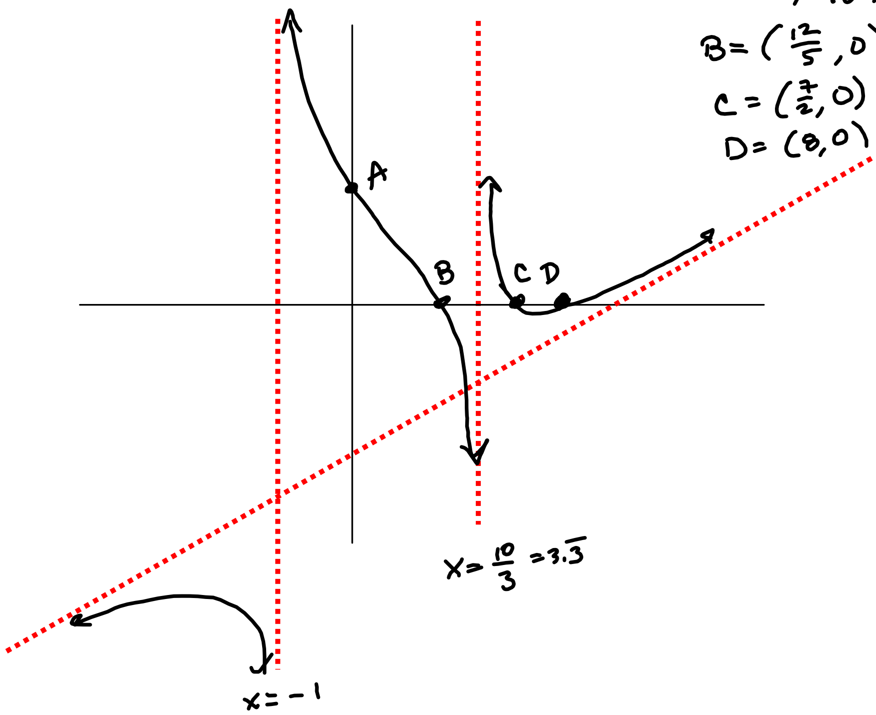


$A = \left(0, \frac{672}{10}\right) = \left(0, \frac{336}{5}\right) = (0, 67.2)$

$B = \left(\frac{12}{5}, 0\right) = (2.4, 0)$

$C = \left(\frac{7}{2}, 0\right) = (3.5, 0)$

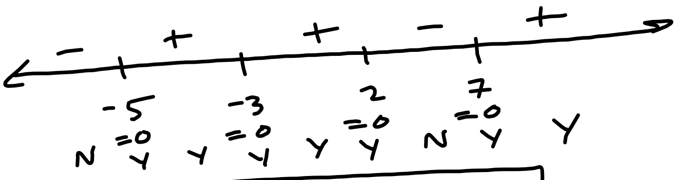
$D = (8, 0)$



11. (2 pts) What is the domain of $W(x)=\sqrt{(x+5)(x-2)^2(x+3)^4(x-7)^5}$?

$W(x) = \sqrt{f(x)}$. Need $f(x) \geq 0$

$(x+5)(x-2)(x+3)^4(x-7)^5 = x^{11} + \dots$

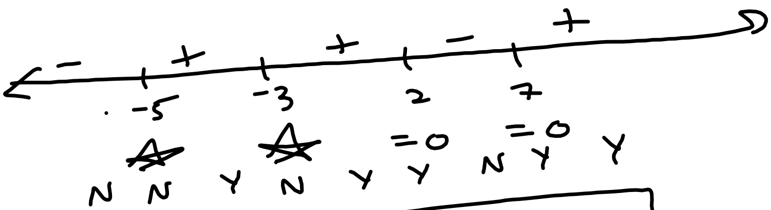


$= \boxed{[-5,2] \cup [7,\infty) = \mathcal{D}(W)}$

12. (2 pts) What is the domain of $K(x)=\sqrt{\dfrac{(x-2)^2(x-7)^5}{(x+5)(x+3)^4}}$?

$K(x) = \sqrt{f(x)}$

Need $f(x) \geq 0$ AND $(x+5)(x+3)^4 \neq 0$



$= \boxed{(-5,-3) \cup (-3,2] \cup [7,\infty) = \mathcal{D}(K)}$