Section 4.3 - Logarithmic Functions

Every exponential function is 1-to-1 and therefore has an inverse function. We define the base *a* logarithm to be this inverse, and write it this way:

$\log_a(x)$. E.g. $\log_3(x)$, $\log_5(x)$

$\log_3(x)$ is the inverse of $f(x) = 3^x$.

$$\boxed{3^{\log_3(x)} = x \qquad \log_3(3^x) = x}$$

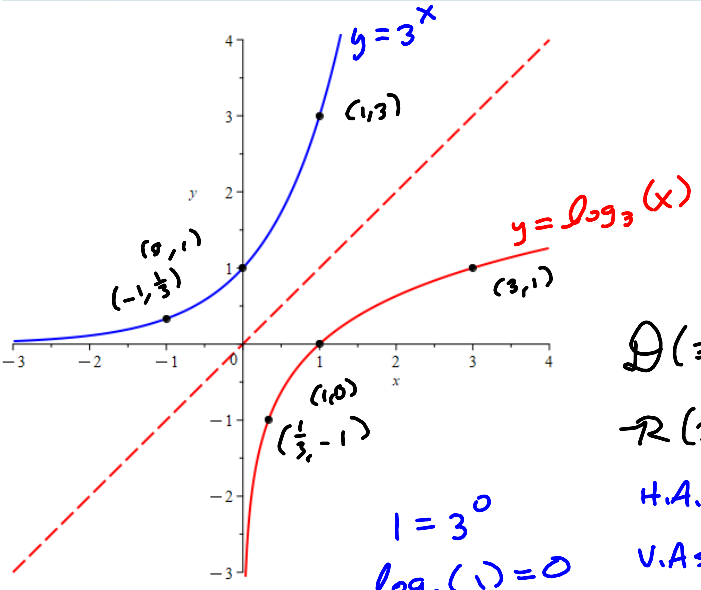
DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base *a***, denoted by $\log_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

SHOW ME YOUR POWER (of 'a')

So $\log_a x$ is the *exponent* to which the base a must be raised to give x.



Logarithm says

"SHOW ME YOUR POWER!"

Logarithms *are* powers.

$\mathcal{D}(3^x) = (-\infty, \infty) = \mathcal{R}(\log_3(x))$

$\mathcal{R}(3^x) = (0, \infty) = \mathcal{D}(\log_3(x))$

H.A. for 3^x is $y = 0$

V.Asymptote for $\log_3(x)$ is $x = 0$.

$1 = 3^0$

$\log_3(1) = 0$

$(a^b)^c \longleftrightarrow \log_2(x^n) = n\log_2(x)$

$a^b a^c = a^{b+c} \longleftrightarrow \log_a(bc) = \log_a(b) + \log_a(c)$

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a (a^x) = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

NATURAL LOGARITHM

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln(x) = \log_e (x)$$

1 $\log(x)$ is the exponent to which the base 10 must be raised to get ---Select--- . So we can complete the following table for $\log(x)$.

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$
$\log(x)$	3	2	1	0	-1	-2	-3	1/2

2 The function $f(x) = \log_{16}(x)$ is the logarithm function with base 16 . So $f(16) =$ 1 , $f(1) =$ 0 , $f\left(\frac{1}{16}\right) =$ -1 , $f(256) =$ 2 , and $f(4) =$ 1/2 .

$4^2 = 16$

$4 = 16^{\frac{1}{2}}$

$\log_{16}(4) = \frac{1}{2}$

3 (a) Convert the following to logarithmic form.

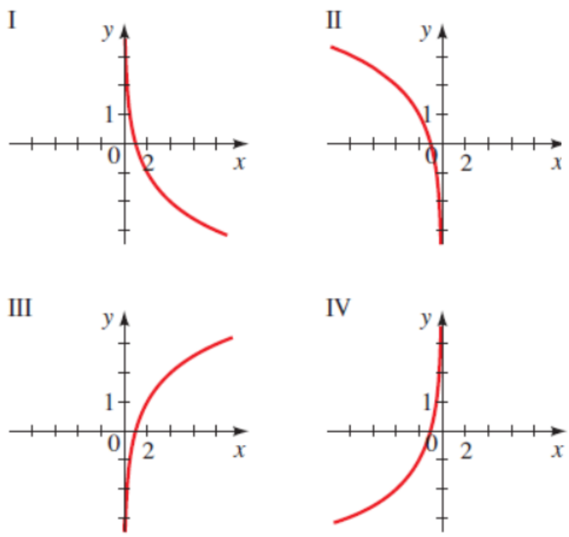
$6^3 = 216 \Longleftrightarrow \log_6(216) = 3$

(b) Convert the following to exponential form.

$\log_6(36) = 2 \Longleftrightarrow 6^2 = 36$

Match the logarithmic function with its graph.

4



(a) $f(x) = -\log_2(x)$

(b) $f(x) = -\log_2(-x)$

(c) $f(x) = \log_2(x)$

(d) $f(x) = \log_2(-x)$

$\log_2(x)$

$D = (0, \infty)$
$R = (-\infty, \infty)$

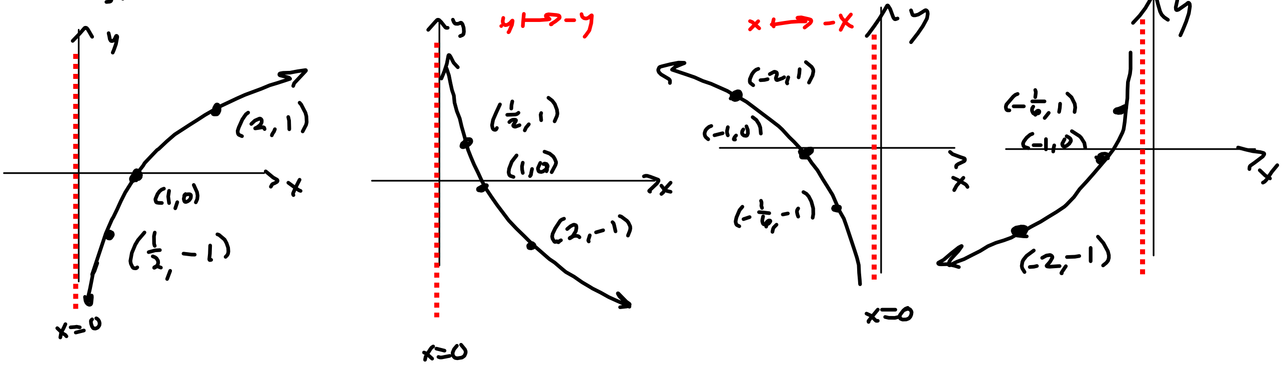
(a) $f(x) = -\log_2(x)$

(b) $f(x) = -\log_2(-x)$

(c) $f(x) = \log_2(x)$

(d) $f(x) = \log_2(-x)$

$\log_2(x)$ $\quad -\log_2(x)$ $\quad \log_2(-x)$ $\quad -\log_2(-x)$



6 The logarithmic function $f(x) = \ln(x - 2)$ has the [---Select---] asymptote $x =$ [2].

vertical

Anywhere its argument is zero!

$\ln(*)$ has vertical asymptote(s) where $* = 0$

$* =$ its argument

$\ln(x^2-4)$ has vertical asymptotes $x=2$, $x=-2$

7 Complete the table by finding the appropriate logarithmic or exponential form of the equation.

Logarithmic Form	Exponential Form
$\log_9(9) = 1$	$9^1 = 9$
$\log_9(729) = 3$	$9^3 = 729$
$\log_9(3) = \frac{1}{2}$	$9^{1/2} = 3$
$\log_9(81) = 2$	$9^2 = 81$
$\log_9\left(\frac{1}{9}\right) = -1$	$9^{-1} = \frac{1}{9}$
$\log_9\left(\frac{1}{81}\right) = -2$	$9^{-2} = \frac{1}{81}$

Write the argument as a power of the base.

729

$3 = 9^{?} = 9^{\frac{1}{2}}$

$81 = 9^2$

$\frac{1}{9} = 9^{-1}$

$= \frac{1}{9^2} = 9^{-2}$

3 | 729
3 | 243
3 | 81
3 | 27
3 | 9
3

9 | 729
9 | 81
9

9 Express the equation in exponential form.

(a) $\log_7(49) = 2$ $\log_7(7^2) = 2$ $7^2 = 49$

(b) $\log_{1/2}(1) = 0$ $(\frac{1}{2})^0 = 1$

$= \log_{1/2}((\frac{1}{2})^0) = 0$

11 Express the equation in exponential form.

(a) $\log_6\left(\frac{1}{216}\right) = -3$ $6^3 = 216 \Rightarrow \frac{1}{216} = \frac{1}{6^3} = 6^{-3}$ $\boxed{6^{-3} = \frac{1}{216}}$

(b) $\log_8(4) = \frac{2}{3}$ $4 = 2^2$ & $8 = 2^3$

$8^{\frac{2}{3}} = 4$

$8 = 2^2 \cdot 2^1 = 4 \cdot 2 = 4 \cdot \sqrt{4} = 4 \cdot 4^{\frac{1}{2}} = 4^{1+\frac{1}{2}}$

$= 4^{\frac{3}{2}} = 8 \Rightarrow$

$(4^{\frac{3}{2}})^{\frac{2}{3}} = 4 = 8^{\frac{2}{3}} \Rightarrow \log_8(4) = \frac{2}{3}$

12 Express the equation in logarithmic form.

(a) $4^2 = 16$ $\log_4(16) = 2$

(b) $10^{-3} = \frac{1}{1,000}$ $\log\left(\frac{1}{1000}\right) = -3$

$\log\left(\frac{1}{10^3}\right) = \log(10^{-3}) = -3$

20 Evaluate the expression. (Simplify your answer completely.)

(a) $e^{\ln(\sqrt{7})}$ If $f(x)=e^x$, then $f^{-1}(x)=\ln(x)=\log_e(x)$

$=(f\circ f^{-1})(\sqrt{7})=\sqrt{7}$

(b) $e^{\ln(1/\pi)}$ $=(f\circ f^{-1})(\frac{1}{\pi})=f(f^{-1}(\frac{1}{\pi}))=f(\ln(\frac{1}{\pi}))=e^{\ln(\frac{1}{\pi})}=\frac{1}{\pi}$

(c) $10^{\log(13)}$ $=13$

21 Evaluate the expression. (Simplify your answer completely.)

(a) $\log_4(0.125)$ $.125=\frac{125}{1000}=\frac{25}{200}=\frac{5}{40}=\frac{1}{8}=\frac{1}{4\cdot 2}=\frac{1}{4\cdot 4^{1/2}}=\frac{1}{4^{1+\frac{1}{2}}}$

$=\frac{1}{4^{3/2}}=4^{-\frac{3}{2}}$

$\log_4(4^{-3/2})=-\frac{3}{2}$

(b) $\ln(e^5)$ $=5$

(c) $\ln(1/e)$ $=\ln(e^{-1})=-1$

22 Evaluate the expression. (Simplify your answer completely.)

(a) $\log_9(\sqrt{3})$ $=\log_9\left(\sqrt{\sqrt{9}}\right)=\log_9\left((9^{\frac{1}{2}})^{\frac{1}{2}}\right)=\log_9\left(9^{\frac{1}{2}(\frac{1}{2})}\right)$

$=\log_9(9^{1/4})=\frac{1}{4}$

(b) $\log_9\left(\frac{1}{3}\right)$ $=\log_9\left(\frac{1}{9^{\frac{1}{2}}}\right)=\log_9(9^{-\frac{1}{2}})=-\frac{1}{2}$

(c) $\log_9(27)$ $=\log_9(9\cdot 3)=\log_9(9\cdot 9^{\frac{1}{2}})=\log_9(9^{1+\frac{1}{2}})=\log_9(9^{\frac{3}{2}})$

$=\frac{3}{2}$

24 Use the definition of the logarithmic function to find x. (Simplify your answer completely.)

(a) $\log_3(x)=-3$ $x=3^{-3}=\frac{1}{3^3}=\frac{1}{27}$

(b) $\log_5(625)=x$

$5^4=(5^2)^2=25^2=625$

$\log_5(625)=\log_5(5^4)=4$

26 Use the definition of the logarithmic function to find x. (Simplify your answer completely.)

(a) $\ln(x) = -9$ $\longrightarrow$ $e^{-9} = x$

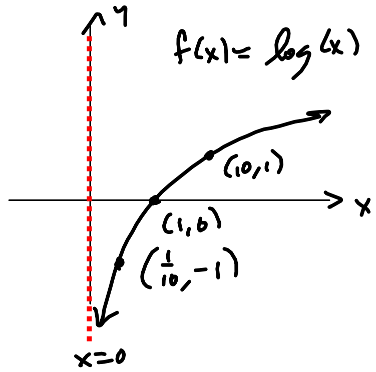
(b) $\ln(1/e) = x$

$\ln(e^{-1}) = x = -1$

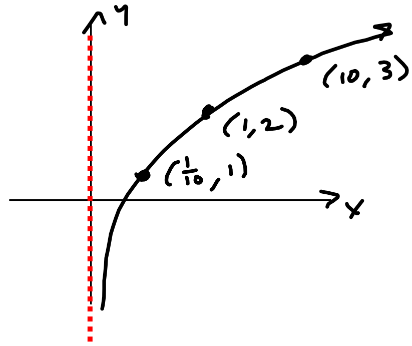
29 Sketch the graph of the function by plotting points.

$g(x) = 2 + \log(x)$

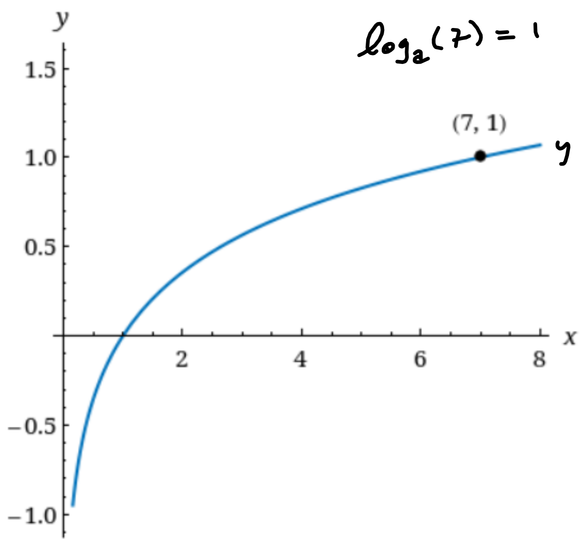
$f(x) = \log(x)$



$g(x) = \log(x) + 2 = f(x) + 2$ $y \longmapsto y+2$



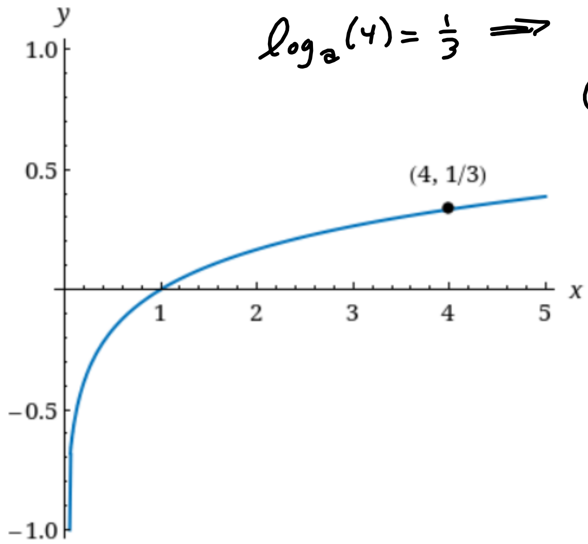
30 Find the function of the form $y = \log_a(x)$ whose graph is given.



$\log_a(7) = 1$ $a^1 = 7 = a$

$y = \log_7(x)$!

32 Find the function of the form $y = \log_a(x)$ whose graph is given.

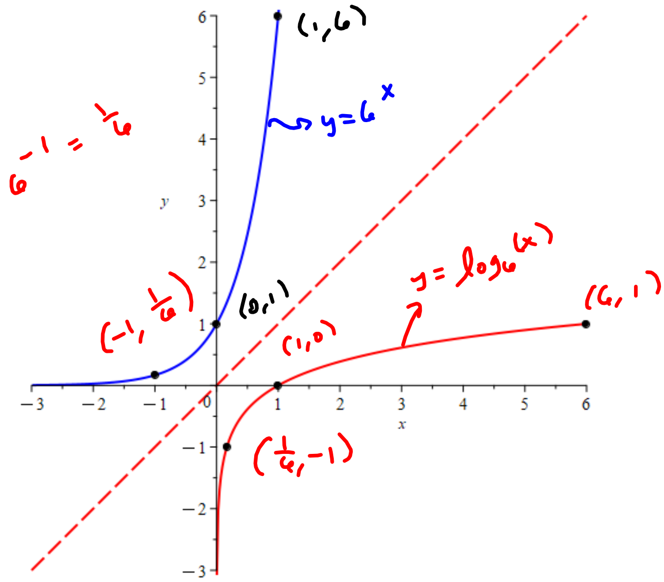


$\log_a(4) = \frac{1}{3} \Rightarrow a^{\frac{1}{3}} = 4$

$(a^{\frac{1}{3}})^3 = 4^3$

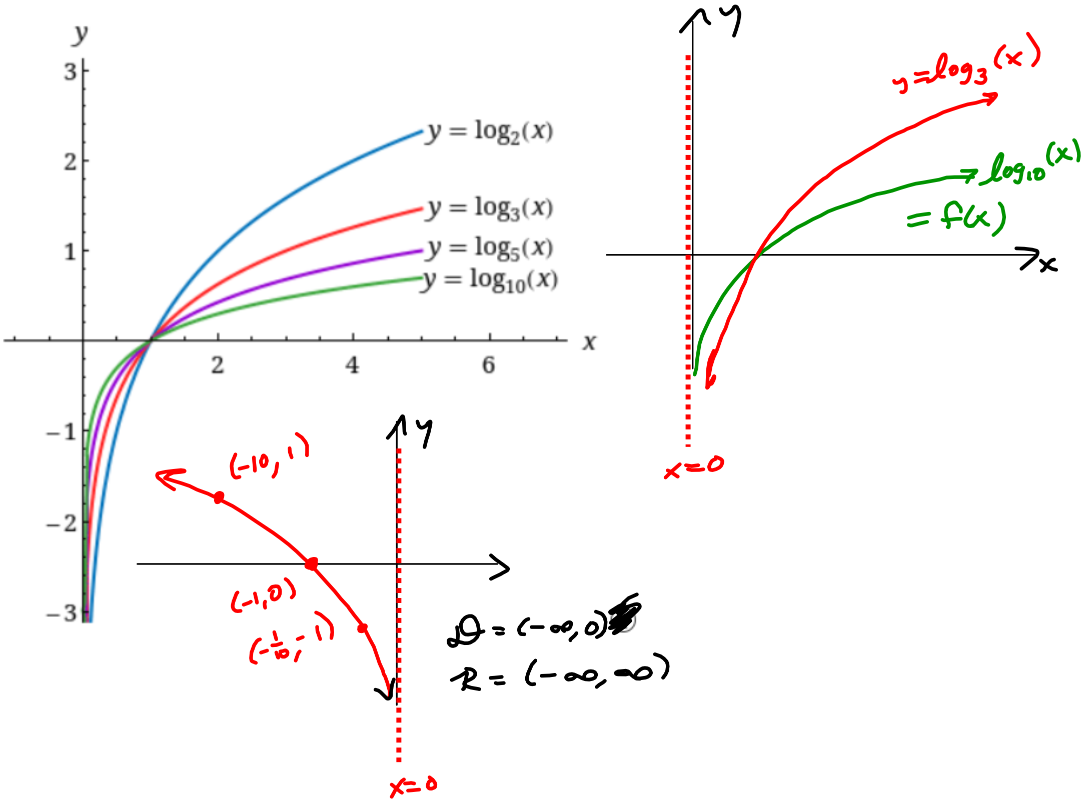
$\boxed{a = 4^3 = 64} \rightarrow y = \log_{64}(x)$

33 Draw the graph of $y = 6^x$, then use it to draw the graph of $y = \log_6(x)$.



35 Graph the function, not by plotting points, but by starting from the graphs in the figures below.

$g(x) = \log_{10}(-x)$



$\mathcal{D} = (-\infty, 0)$

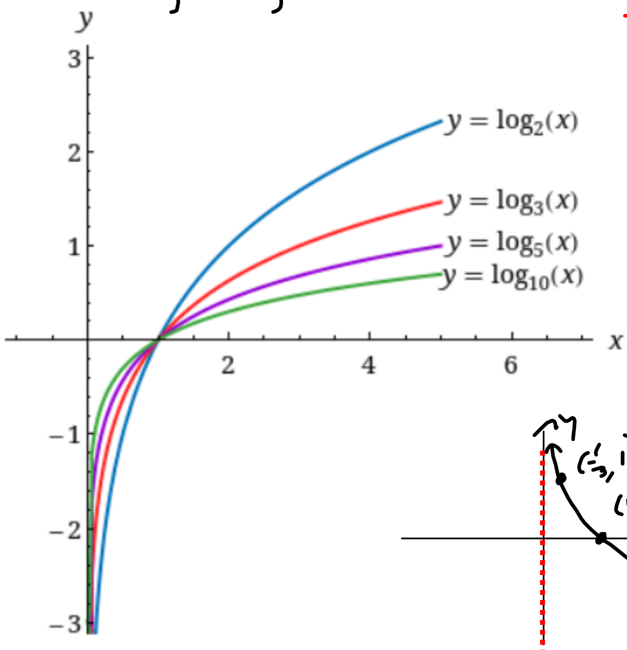
$\mathcal{R} = (-\infty, \infty)$

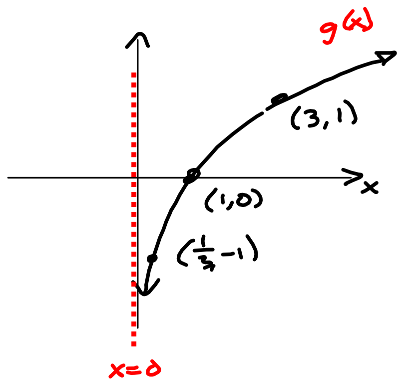
36 Graph the function, not by plotting points, but by starting from the graphs in the figures below.

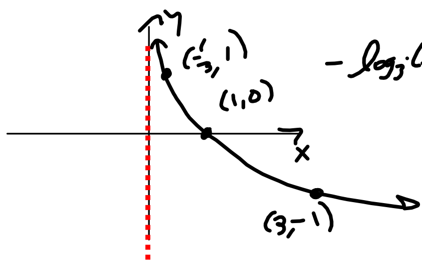
$f(x) = -\log_3(x) = -g(x)$ **State the domain and range.**

$y \mapsto -y$

$g(x) = \log_3(x) =$ Basic Function







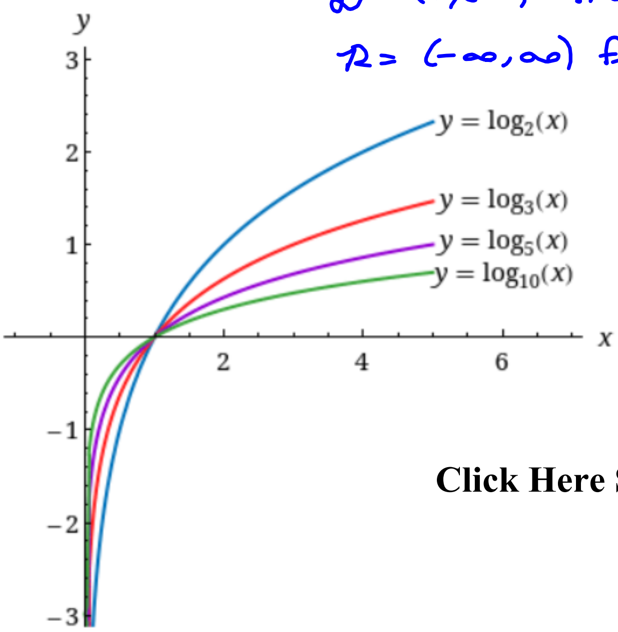
$-\log_3(x) = f(x)$
$= -g(x)$

Graph the function, not by plotting points, but by starting from the graphs in the figures below.

37 $f(x) = \log_2(x - 2)$

$\mathcal{D} = (2, \infty)$ from "Need $x - 2 > 0$."

$\mathcal{R} = (-\infty, \infty)$ from graph.



Click Here See Video for #s 37, 38.

38 Graph the function, not by plotting points, but by starting from the graphs in the figures below. (Select Update Graph to see your response plotted on the screen. Select the Submit button to grade your response.)

$g(x) = \ln(x + 3)$

Demonstrate the Graphing Tool on WebAssign.

State Domain, Range, and Vertical Asymptote.

$\mathcal{D} = (-3, \infty)$

$\mathcal{R} = (-\infty, \infty)$

42 Graph the function, not by plotting points, but by starting from the graphs in the figures below. (Select Update Graph to see your response plotted on the screen. Select the Submit button to grade your response.)

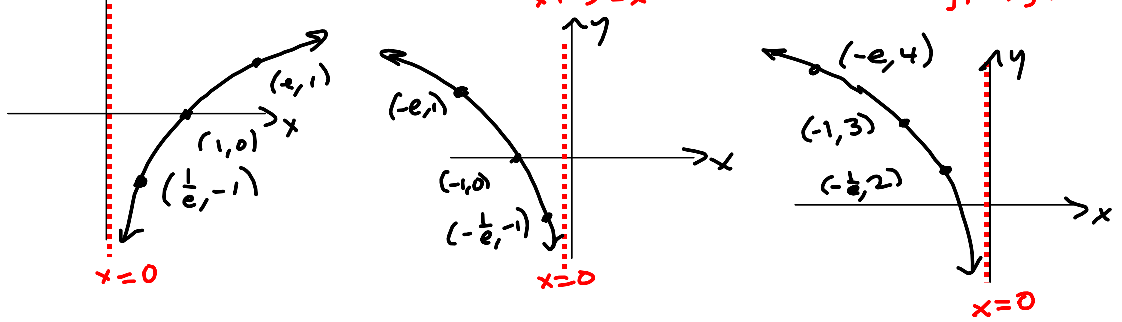
$y = 3 + \ln(-x)$

Demonstrate the WebAssign Graphing Tool.

State the Domain, Range, and Vertical Asymptote.

$f(x) = \ln(x)$ $\qquad \ln(-x) = f(-x)$ $\qquad \ln(-x) + 3 = f(-x) + 3$

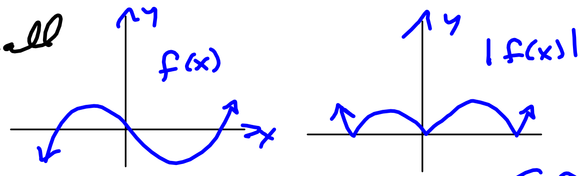
$x \mapsto -x$ $\qquad y \mapsto y+3$



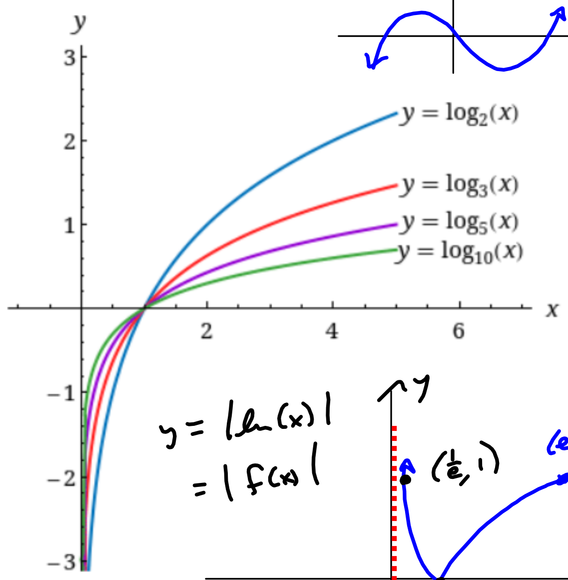
Graph the function, not by plotting points, but by starting from the graphs in the figures below.

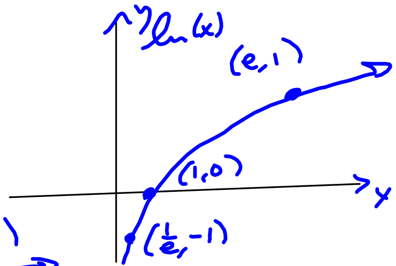
43 $y = |\ln(x)|$

Recall

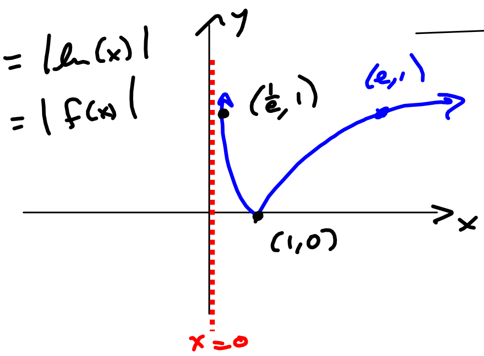


$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$





$y = |\ln(x)|$
$= |f(x)|$



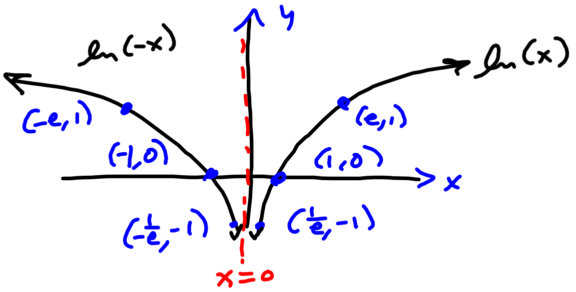
Graph the function, not by plotting points, but by starting from the graphs in the figures below.

44 $y = \ln(|x|)$

Need $|x| > 0$ That means $x \neq 0$

$|x| = \begin{cases} x \text{ if } x \geq 0 & \text{is always} \geq 0 \text{ Throw out } x=0 \\ -x \text{ if } x < 0 & \text{is always} > 0\,! \end{cases}$

So $\mathcal{D} = (-\infty, 0) \cup (0, \infty)$
$= \mathbb{R} \setminus \{0\}$



$$\ln(|x|) = \begin{cases} \ln(x) & \text{if } x \geq 0 \\ \ln(-x) & \text{if } x < 0 \end{cases} = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

Need $|x| \neq 0$, so

Find the domain of the function. (Enter your answer using interval notation.)

45 $g(x) = \log_4(x^2 - 9) = \log_4((x-3)(x+3))$

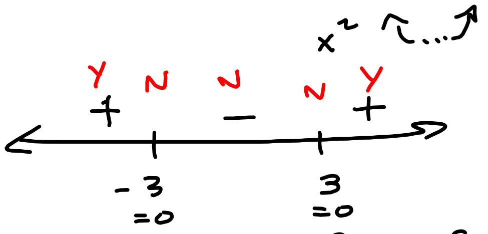
Domain:

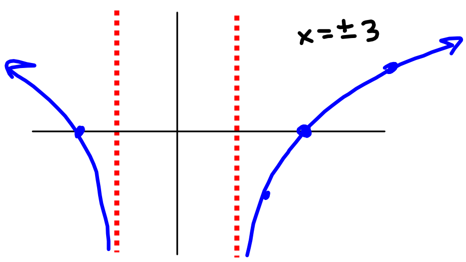
Need $x^2 - 9 > 0$

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$





Test $x = 0$. $0^2 - 9 = -9 < 0$ No "−"

$$\boxed{\mathcal{D} = (-\infty, -3) \cup (3, \infty)}$$

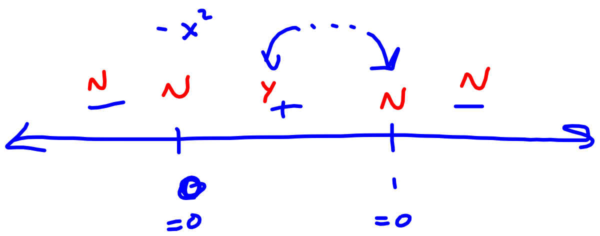
Find the domain of the function. (Enter your answer using interval notation.)

46

$g(x) = \ln(x - x^2) = \ln(x(1-x))$

Need $x - x^2 > 0$

$x - x^2 = 0 \Rightarrow x = 0, 1$



$\boxed{D = (0,1)}$

47 Find the domain of the function. (Enter your answer using interval notation.)

$h(x) = \ln(x) + \ln(6 - x)$

Keep both arguments positive

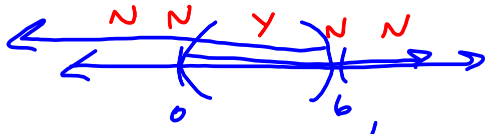
$\mathcal{D}(f \pm g) = \mathcal{D}(f) \cap \mathcal{D}(g)$

$= \{x \mid x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\}$

Need $x > 0$
$(0, \infty)$

AND

Need
$6 - x > 0$
$6 > x$
$x < 6$
$(-\infty, 6)$



$x > 0$ AND $x < 6$

AND Means BOTH!

$= \boxed{(0, 6) = \text{Domain}}$

48 Find the domain of the function. (Enter your answer using interval notation.)

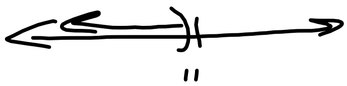
$h(x) = \sqrt{x - 3} - \log_5(11 - x)$

Need $x - 3 \geq 0$
$x \geq 3$

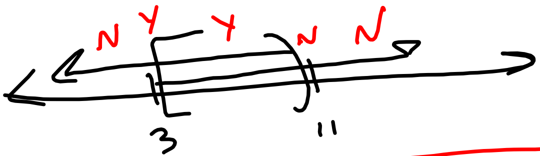
$[3, \infty)$

AND

& $11 - x > 0$
$11 > x$
$x < 11$



$(-\infty, 11)$



AND : $\mathcal{D}(\sqrt{x-3}) \cap \mathcal{D}(\log_5(11-x))$

$\boxed{= [3, 11) = \mathcal{D}(h(x))}$

A graphing device is recommended.

Draw the graph of the function in a suitable viewing rectangle.

51 $y = \dfrac{\ln(x)}{x}$ $\mathcal{D}: x \neq 0$ $\to x > 0$ $\{x \mid x > 0\} = \mathcal{D} = (0, \infty)$

Use the graph to find the domain, the asymptotes, and the local maximum and minimum values. (Enter your answer using interval notation. Round your answers to two decimal places. Enter your answers as a comma-separated list. If an answer does not exist, enter DNE.)

domain $(0, \infty)$

vertical asymptote $x = 0$

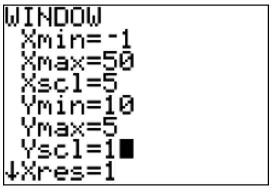
horizontal asymptote $y = 0$

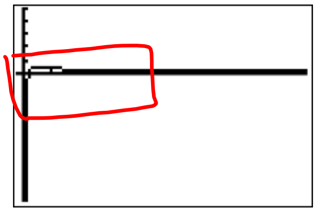
local maximum $(2.72, .37)$

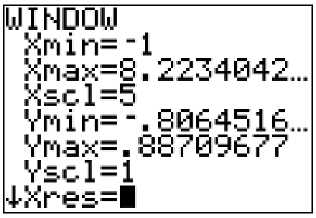
local minimum DNE

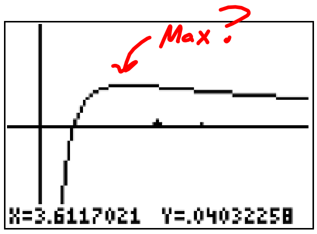
X	Y_1
.001	-6908
1E-4	-92103
1000	.00691
10000	9.2E-4
1E8	1.8E-7

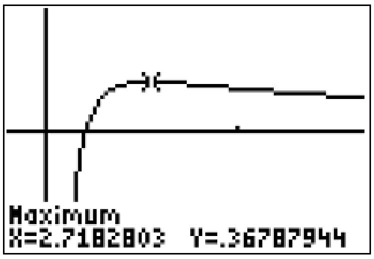
X=











Local max of $y \approx 0.36787944$ @ $x \approx 2.7182803$.