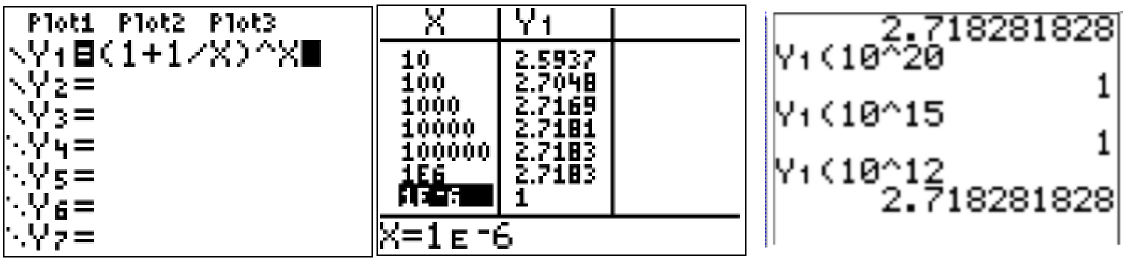
Section 4.2 - The Natural Logarithm Function

Definition: The number *e* is defined by the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$



Compound Interest Formula:

$$P\left(1+\frac{r}{m}\right)^{mt} = P\left(1+\frac{1}{\left(\frac{m}{r}\right)}\right)^{m\left(\frac{r}{r}\right)t} = P\left(1+\frac{1}{\left(\frac{m}{r}\right)}\right)^{\frac{m}{r}(rt)}$$

$$= P\left(\left(1+\frac{1}{\left(\frac{m}{r}\right)}\right)^{\frac{m}{r}}\right)^{rt}$$

If $r>0$ is fixed, then

$$\lim_{m\to\infty}\frac{m}{r} = \frac{1}{r}\lim_{m\to\infty} m = \frac{1}{r}\cdot\infty = \infty$$

& so, $\displaystyle\lim_{m\to\infty} P\left(\left(1+\frac{1}{\left(\frac{m}{r}\right)}\right)^{\frac{m}{r}}\right)^{rt} = P\left(\lim_{\frac{m}{r}\to\infty}\left(1+\frac{1}{\left(\frac{m}{r}\right)}\right)^{\frac{m}{r}}\right)^{rt} = Pe^{rt}$!

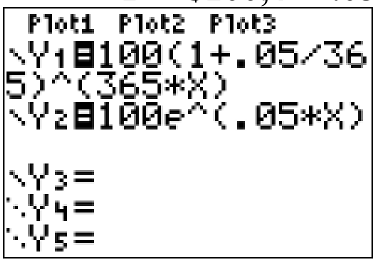
Continuous Compounding:

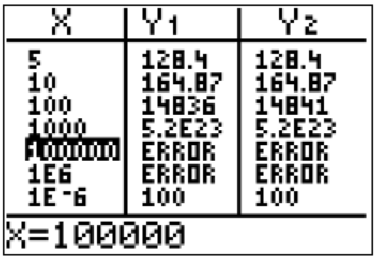
$A(t) = Pe^{rt}$, where e = the natural base

'e' for 'Euler" who discovered it.

Continuous Compounding is very close to Daily Compounding:

***P* = $100, *r* = .05, compounded daily and continuously, side by side.**

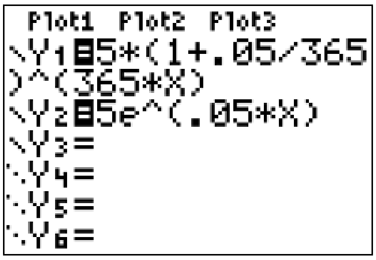


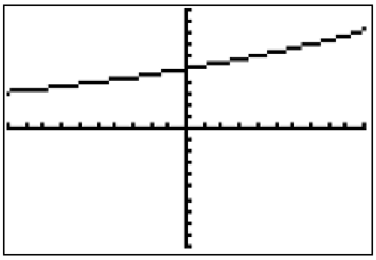


X	Y1	Y2
5	128.4	128.4
10	164.87	164.87
100	14836	14841
1000	5.2E23	5.2E23
100000	ERROR	ERROR
1E6	ERROR	ERROR
1E-6	100	100

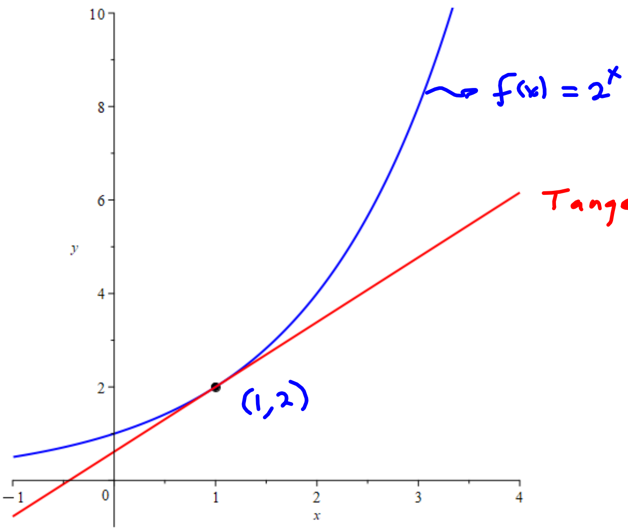
X=100000

***P* = $5 *r* = .05, compounded daily and continuously, side by side in a 10x10 square window. I used $5 so you could see both of them in the standard window. According to my TI-84, the two are one, graphically.**



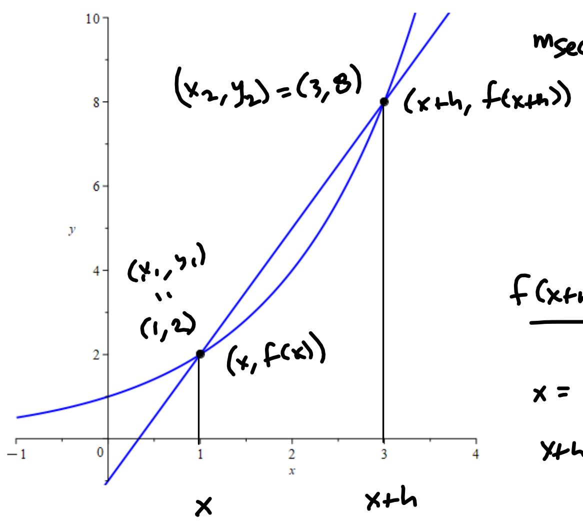


Calculus Connection. How do we find the slope of $f(x)=2^x$ at $x = 1$?



Tangent Line,
what's its slope?
All we know is slope of a straight line and the average slope of a function over an interval.

In College Algebra, we know how to do a secant line:



$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{3-1} = \frac{6}{2} = 3$$

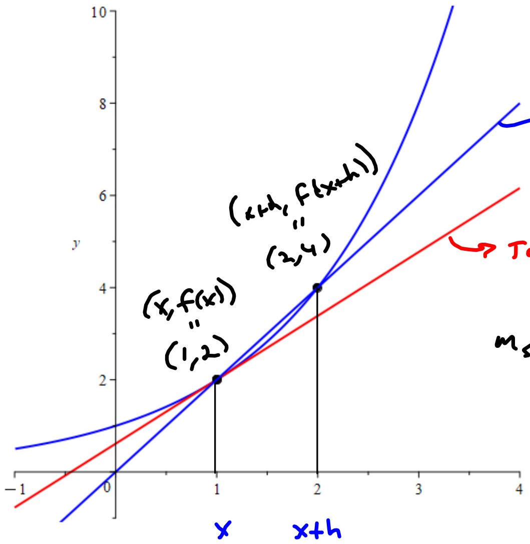
Calculus takes this idea and says "Let's take the 2nd point closer and closer to the first."

$$\frac{f(x+h) - f(x)}{h} = \frac{f(3) - f(1)}{3-1} = \dots = 3$$

$x = 1,\ h = 2$

$x + h = 3.$

We can take the 2nd point closer to $x_1 = 1$, say, $x_2 = 2$.



$m_{sec} = \frac{4-2}{2-1} = \frac{2}{1} = 2$

Tangent slope = ?

$m_{sec} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{2^{x+h} - 2^x}{h}$

$= \frac{2^x 2^h - 2^x}{h} = \frac{2^x(2^h - 1)}{h} = \frac{2^h - 1}{h} \cdot 2^x = \frac{2^h-1}{h} f(x)$!

and the idea in calculus is to let $h \to 0$, so they look at

$g(h) = \frac{2^h - 1}{h}$, as $h \to 0$, which we have the technology to explore numerically:

```
Plot1 Plot2 Plot3
\Y1=2
\Y2=(Y1^X-1)/X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

X	Y1	Y2
.01	2	.69556
.001	2	.69339
1E-5	2	.69315
1E-6	2	.69315

X=

This says that the slope of $f(x) = 2^x$ is approximately $(.69315) \cdot f(x)$

$= .69315 \cdot 2^x$

Try: 3^x

slope $\approx (1.0986)3^x$

X	Y_1	Y_2
.01	3	1.1047
.001	3	1.0992
1E-5	3	1.0986
1E-6	3	1.0986

X=.01

Find b so that the slope of b^x is $1 \cdot b^x = b^x$

$b > 2$, $b < 3$

$b = 2.5$? $.9169 \cdot 2.5^x$

$b = 2.75$ 1.0116

X	Y_1	Y_2
1E-7	2.72	1.0006
.001	2.72	1.0011
1E-5	2.72	1.0006
1E-6	2.72	1.0006

X=1E-7

X	Y_1	Y_2
1E-7	2.71	.99695
.001	2.71	.99745
1E-5	2.71	.99695
1E-6	2.71	.99695

X=1E-7

X	Y_1	Y_2
1E-7	2.715	.99879
.001	2.715	.99929
1E-5	2.715	.9988
1E-6	2.715	.99879

X=1E-7

X	Y_1	Y_2
1E-7	2.718	.9999
.001	2.718	1.0004
1E-5	2.718	.9999
1E-6	2.718	.9999

X=1E-7

X	Y_1	Y_2
1E-7	2.719	1.0003
.001	2.719	1.0008
1E-5	2.719	1.0003
1E-6	2.719	1.0003

X=1E-7

X	Y_1	Y_2
1E-7	2.7183	1
.001	2.7183	1.0005
1E-5	2.7183	1
1E-6	2.7183	1

X=1E-7

1 The function $f(x) = e^x$ is called the ---Select--- exponential function. The number e is approximately equal to [ ] . (Round your answer to five decimal places.)

2.71828 natural

compounded

2 In the formula $A(t) = Pe^{rt}$ for continuously compound interest, the letters P, r, and t stand for ---Select--- , ---Select--- , and ---Select--- , respectively, and $A(t)$ stands for ---Select--- . So if \$300 is invested at an interest rate of 8% compounded continuously, then the amount after 2 years is \$ [ ] . (Round your answer to the nearest cent.)

A(t)= future value as a function of t.

P= principal, annual percentage rate=r, t=time, in years.

$Pe^{rt} = 300e^{.08 \cdot 2}$

```
                1.0024689
Y2(.000001
                1.0006324
e^(1
              2.718281828
300e^(.05*2)
              331.5512754
```

≈ \$331.55

3 Complete the table of values, rounded to two decimal places.

x	$f(x) = 1.1e^x$
−2	
−1	
−0.5	
0	
0.5	
1	
2	

Dreadfully tedious.

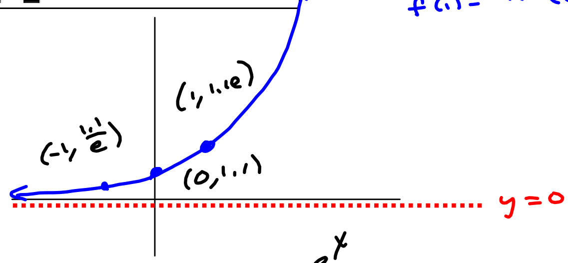
```
 Plot1 Plot2 Plot3
\Y1=1.1e^(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
   X   | Y1
  -2   | .14887
  -1   | .40467
  -.5  | .66718
  0    | 1.1
  .2   | 1.3435
  1    | 2.9901
  2    | 8.128
X=2
```

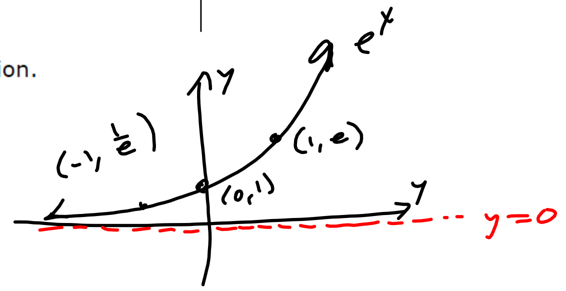
$f(-1) = 1.1(\frac{1}{e})$

$f(0) = 1.1(1)$

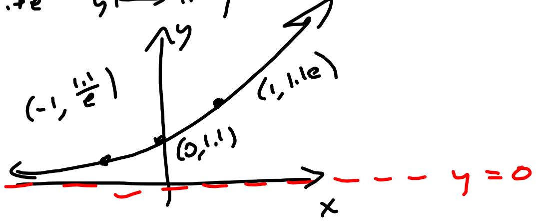
$f(1) = 1.1(e)$



Sketch a graph of the function.

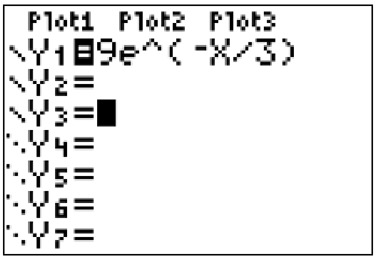


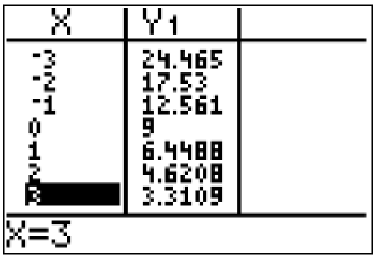
$1.1e^x$ $y \mapsto 1.1y$



4 Complete the table of values, rounded to two decimal places.

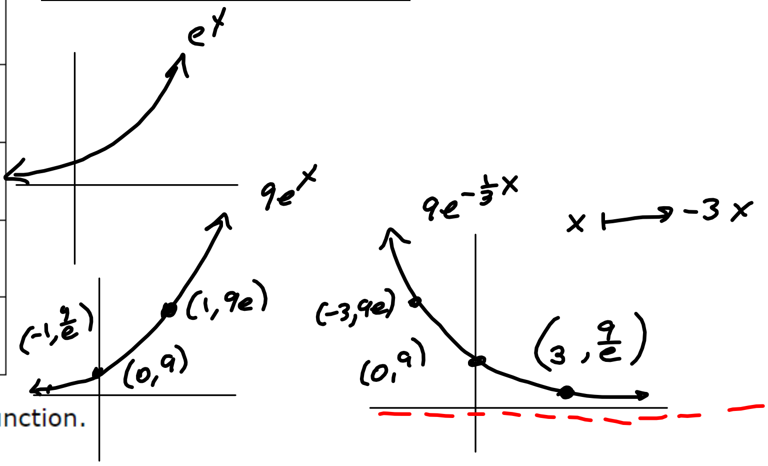
x	$f(x) = 9e^{-x/3}$
-3	
-2	
-1	
0	
1	
2	
3	





X	Y1
-3	24.465
-2	17.53
-1	12.561
0	9
1	6.4488
2	4.6208
3	3.3109

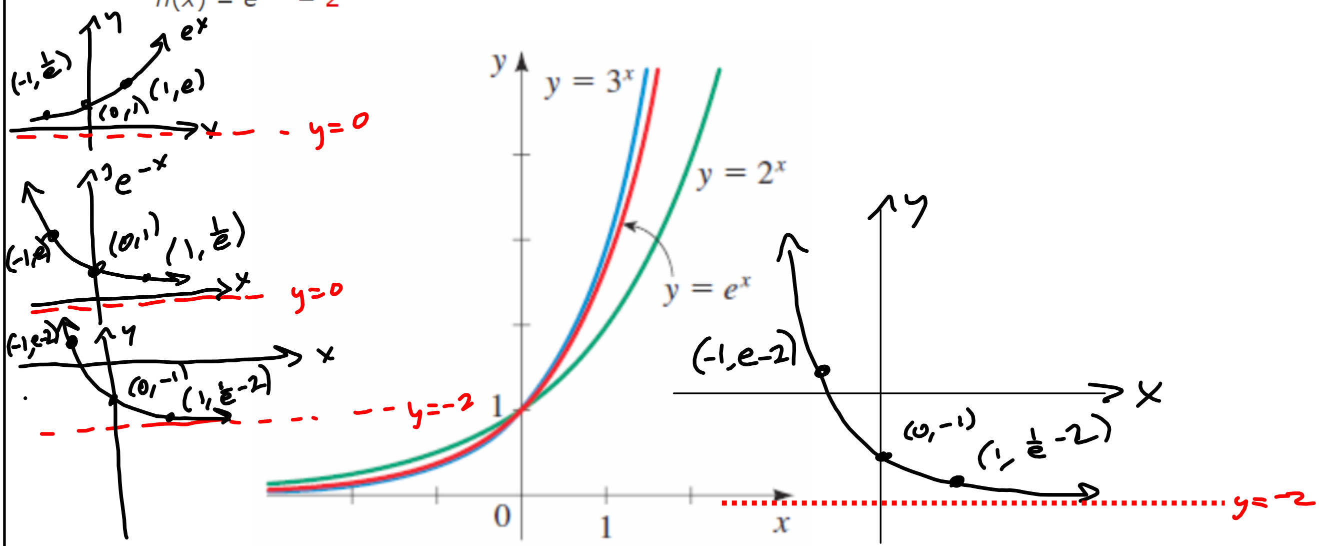
X=3



Sketch a graph of the function.

5 Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in the figure below.

$h(x) = e^{-x} - 2$



State the domain and range. (Enter your answers using interval notation.)

$\mathcal{D} = (-\infty, \infty) = \mathbb{R}$

$\mathcal{R} = (-2, \infty)$

State the asymptote.

H.A.: $y = -2$

Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in the figure below.

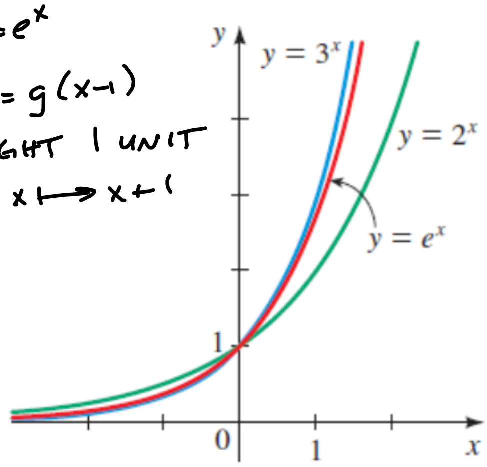
6 $f(x) = e^{x-1}$

$g(x) = e^x$

& $f(x) = g(x-1)$

RIGHT 1 UNIT

$x \mapsto x+1$



State the domain and range. (Enter your answers using interval notation.)

State the asymptote.

Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in the figure below.

7 $y = e^{x-1} + 4$

$$f(x) = e^x \xrightarrow[x \mapsto x+1]{\text{RIGHT } 1} e^{x-1} \xrightarrow[y \mapsto y+4]{\text{UP } 4} e^{x-1} + 4$$

State the y-intercept, domain, range, and horizontal asymptote. (Enter your answers using interval notation. Round your intercept to two decimal places.)

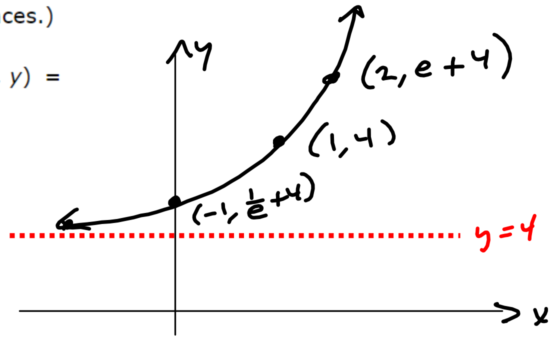
y-intercept $(x, y) =$

domain $\mathbb{R}$

range $(4, \infty)$

horizontal asymptote

$y = 4$



8 Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in the figure below.

$$h(x) = e^{x+1} - 4$$

State the y-intercept, domain, range, and horizontal asymptote. (Enter your answers using interval notation. Round your intercept to two decimal places.)

y-intercept $(x, y) =$

domain

range

horizontal asymptote

Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in the figure below.

9 $$g(x) = -e^{x-1} - 4$$

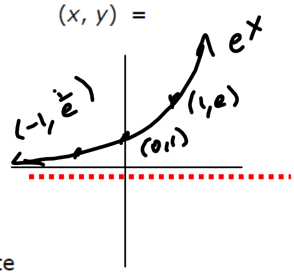
State the y-intercept, domain, range, and horizontal asymptote. (Enter your answers using interval notation. Round your intercept to two decimal places.)

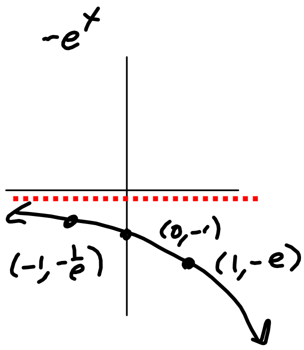
y-intercept $(x, y) =$

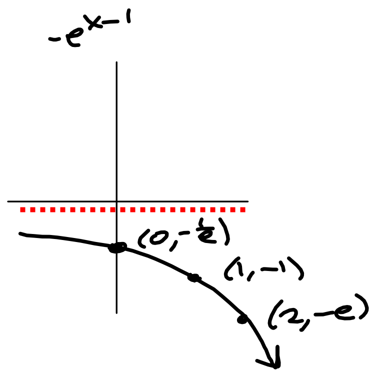
domain

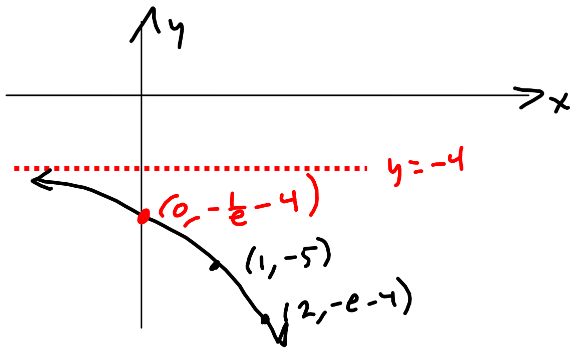
range

horizontal asymptote





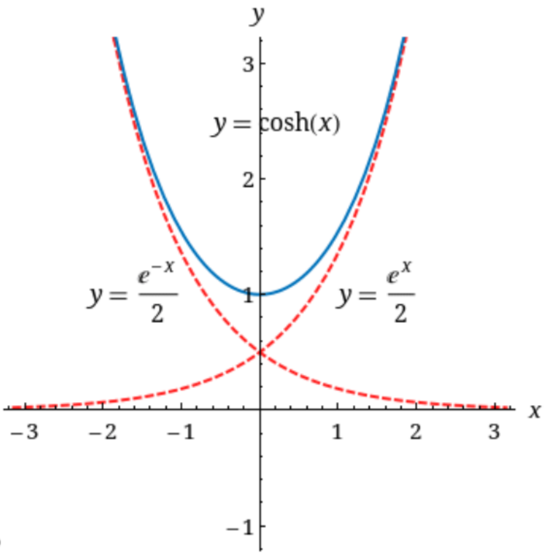


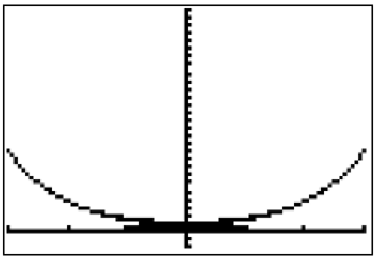


The *hyperbolic cosine function* is defined by

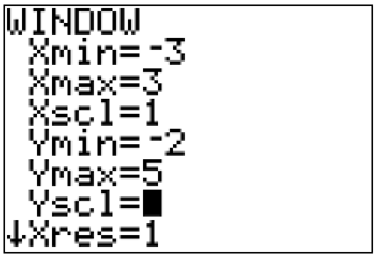
10 $$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

(a) Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes, and use graphical addition to sketch the graph of $y = \cosh(x)$.





Graphing calculator lacks fidelity to show where one leaves offf and the other begins. Shrinking down the vertical:



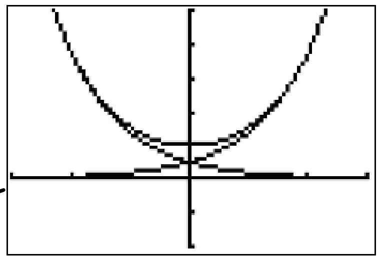
(b) Use the definition to show that $\cosh(-x) = \cosh(x)$.

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{+x}}{2}$$

$$= \frac{e^{x} + e^{-x}}{2} = \cosh(x) \rightarrow$$

cosh(x) is even

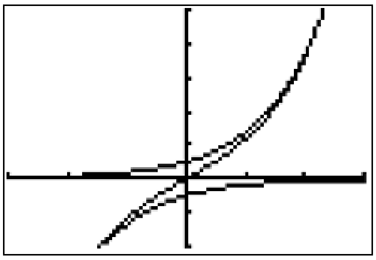
$f(-x) = f(x)$



The *hyperbolic sine function* is defined by

11 $$\sinh(x) = \frac{e^x - e^{-x}}{2}. = \frac{e^x + (-e^{-x})}{2}$$

(a) Sketch the graphs of the functions $y = \frac{e^x}{2}$ and $y = -\frac{e^{-x}}{2}$ on the same axes, and use graphical addition to sketch the graph of $y = \sinh(x)$.



(b) Use the definition to show that $\sinh(-x) = -\sinh(x)$.

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{x} = \frac{e^{-x} - e^x}{2}$$

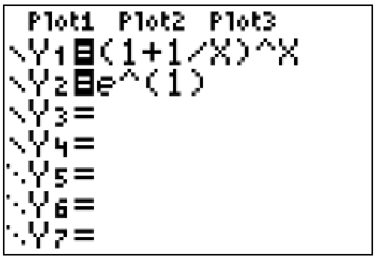
$$= -\frac{e^x - e^{-x}}{2} = -f(x), \text{ i.e.,}$$

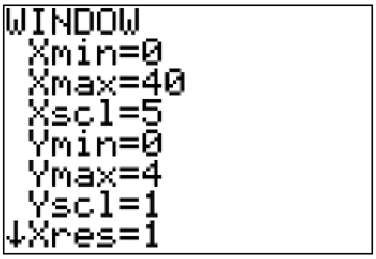
$f(x)$ is odd.

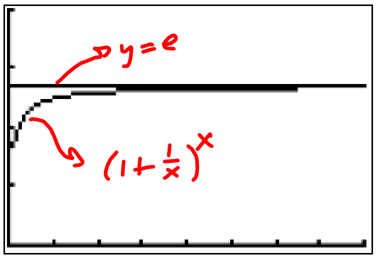
12

A graphing device is recommended.

Illustrate the definition of the number e by graphing the curve $y = (1 + 1/x)^x$ and the line $y = e$ on the same screen, using the viewing rectangle [0, 40] by [0, 4].







A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

13 $m(t) = 11e^{-0.016t}$ $m(0) = m_0 = 11e^{-0} = 11$

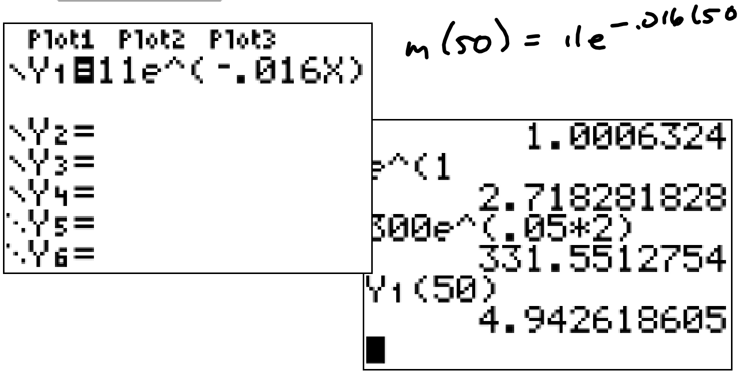
where $m(t)$ is measured in kilograms.

(a) Find the mass (in kg) at time $t = 0$.

11 kg

(b) How much of the mass (in kg) remains after 50 days? (Round your answer to one decimal place.)

kg



$m(50) = 11e^{-.016(50)} \approx 4.942618605$ Kg

$\approx \boxed{4.9 \text{ Kg}}$

An investment of $3,000 is deposited into an account for which interest is compounded continuously. Complete the table by filling in the amounts to which the investment grows at the indicated times. (Round your answers to the nearest cent.)

14

r = 6%

Time (years)	Amount
1	$
2	$
3	$
4	$
5	$
6	$

	A	B	C	D	E	F	G
1	**Section 4.2 #14**			**CONTINOUS COMPOUNDING:**			
2							
3					$3,185.51		
4	*P*	*r*			$3,382.49		
5	$3,000.00	0.06			CONTINOU=A5*EXP(B5*B10)		
6							
7		*t*	*A* (*t*)		*t* = time in years		
8		1	$3,185.51		*r* = Annual Percentage Rate		
9		2	$3,382.49		*A* = Future Value		
10		3	$3,591.65		*P* = Principal Amount or		
11		4	$3,813.75		*present value*		
12		5	$4,049.58		*e* = the *natural* exponential base		
13		6	$4,299.99				
14		7	$4,565.88				

If $3,000 is invested at an interest rate of 4.5% per year, compounded continuously, find the value of the investment after the given number of years. (Round your answers to the nearest cent.)

15

(a) 2 years

$

(b) 4 years

$

(c) 12 years

$

	A	B	C	D	E	F	G	H	I
1	**Section 4.2 #15**				CONTINUOUS COMPOUNDING: $A(t)=Pe^{rt}$				
2									
3					$3,138.08				
4	*P*	*r*			$3,282.52				
5	$3,000.00	0.045			$3,433.61				
6									
7		*t*	*A* (*t*)		*t* = time in years				
8		1	$3,138.08		*r* = Annual Percentage Rate				
9		2	$3,282.52		*A* = Future Value				
10		3	$3,433.61		*P* = Principal Amount or				
11		4	$3,591.65		*present value*				
12		12	$5,148.02		*e* = the *natural* exponential base				
13		13	$5,384.97						
14		14	$5,632.83						

Sheet1

16 A graphing device is recommended.

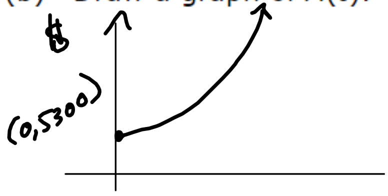
A sum of \$5,300 is invested at an interest rate of 9% per year, compounded continuously.

(a) Find the value $A(t)$ of the investment after t years.

$A(t) = \boxed{Pe^{rt}}$

P = Principal Amt or Present Value, in \$
e = natural base
r = annual % rate as a decimal.
t = time in yrs.

(b) Draw a graph of $A(t)$.

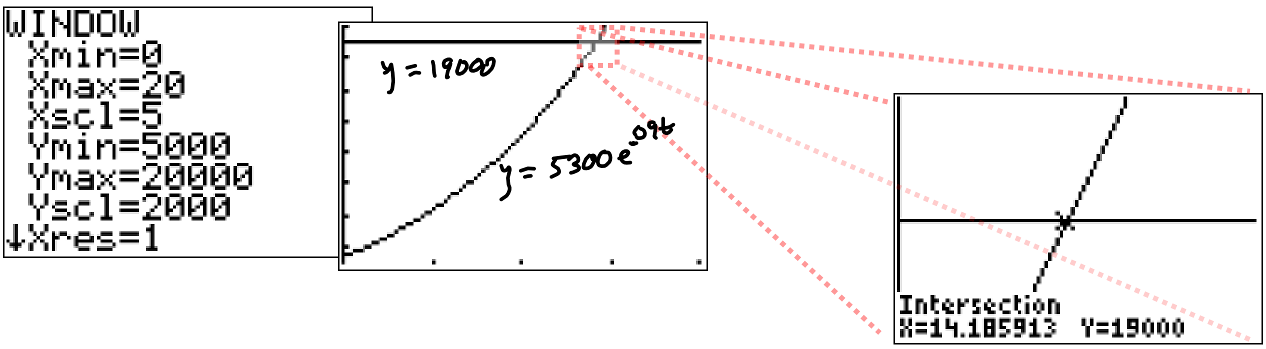


(c) Use the graph of $A(t)$ to determine when (in yr) this investment will amount to \$19,000. (Round your answer to the nearest whole number.)

[ ] yr

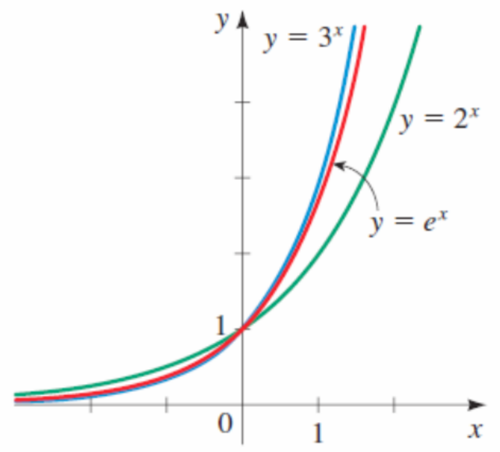
Solve $A(t) = 5300e^{.09t} = \$19000$

Graph $\Rightarrow$ $t \approx 14.185913$ yrs
$\approx$ 14 yrs



Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in the figure below.

17 $g(x) = 3 + e^x$



State the domain and range. (Enter your answers using interval notation.)

domain

range

State the asymptote.