3.7 - Polynomial and Rational Inequalities

1 To solve a polynomial inequality, we factor the polynomial into irreducible factors and find all the real [---Select---] of the polynomial. Then we find the intervals determined by the real [---Select---] and use test points in each interval to find the sign of the polynomial on that interval. Let

$P(x) = x(x + 4)(x - 1)$. $= x^3 + \sim$ $-$ $+$

Fill in the table below to find the intervals on which $P(x) \geq 0$.

The quick way: Use general understanding of end behavior and sign changes.

N Y Y Y N Y Y
$-$ $+$ $-$ $+$
-4 0 1
$=0$ $=0$ $=0$

≥ 0 "+" "=0"

$\Rightarrow x \in [-4, 0] \cup [1, \infty)$

Book wants / teacher

Sign of	$(-\infty, -4)$	$(-4, 0)$	$(0, 1)$	$(1, \infty)$
x	$-$	$-$	$+$	$+$
$x + 4$	$-$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$+$
$x(x + 4)(x - 1)$	$-$	$+$	$-$	$+$

-4 0 1 $\to x$

Alternate Test Values

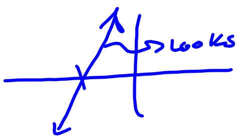
	Test	
$(-\infty, -4)$	$x = -5$	
$(-4, 0)$	$x = -1$	$P(-1) = (-1)(-1+4)(-1-1) = -(3)(-2) = +6$ $+$
$(0, 1)$	$x = \frac{1}{2}$	$P(\frac{1}{2}) = (\frac{1}{2})(\frac{1}{2}+4)(\frac{1}{2}-1) = -$
$(1, \infty)$	$x = 2$	$P(2) = (2)(2+4)(2-1) = +$

$-$ $+$ $-$ $+$
-4 0 1

One way to analyze these.

(a) $x = -4$, $x + 4 = 0$ & $x = -4$, $x - 1 = -5$
So near $x = -4$, but not $x = -4$, we have

$x(x+4)(x-1) \approx (-4)(x+4)(-5) \approx 20(x+4)$



$\to$ looks like $y = 20(x+4)$ near $x = -4$

To solve a rational inequality, we factor the numerator and the denominator into irreducible factors. The cut points are the real ---Select--- of the numerator and the real ---Select--- of the denominator. Then we find the intervals determined by the ---Select--- , and we use test points to find the sign of the rational function on each interval. Let

2

$$r(x) = \frac{(x+5)(x-1)}{(x-7)(x+9)} = \frac{x^2 + \cdots}{x^2 + \cdots} \quad x \to \pm \text{BIG} \longrightarrow \frac{x^2}{x^2} = \boxed{1 = y \text{ is H.A.}}$$

Fill in the diagram below to find the intervals on which $r(x) \geq 0$.

y-int (as a check)

$$r(0) = \frac{5(-1)}{-7(9)} = \frac{-5}{-63} = \frac{5}{63} \checkmark$$

"+"

Cut points:

$x = -5, 1$ "= 0"

$x = -9, 7$ "$\nexists$"

Y N N Y Y Y N N Y

\+ − + ✓ − +

-9 ($\nexists$), -5 ($=0$), 1 ($=0$), 7 ($\nexists$)

≥ 0 "= 0" OR "+"

$\longrightarrow x \in \boxed{(-\infty, -9) \cup [-5, 1] \cup (7, \infty)}$

Solve the inequality. (Enter your answer using interval notation.)

5

$(x + 6)^2(x + 2)(x - 1) > 0$

$x^4 + \cdots$

\+ ↖ ... ↗ +

−6

Y N Y N N N Y

\+ + − +

-6 ($=0$), -2 ($=0$), 1 ($=0$)

"> 0" "+"

$\longrightarrow x \in \boxed{(-\infty, -6) \cup (-6, -2) \cup (1, \infty)}$

Solve the inequality.

6

$f(x) = (4x-9)^4(x-1)^3(x+1) \le 0 \longrightarrow f(x) = (4x)^4(x^3)(x^1) + \text{smaller stuffs}$

zeros: $\frac{9}{4}, 1, -1$ $\qquad = 4^4 x^8 \qquad x^8 \; \nwarrow \ldots \nearrow$

mult $4, 3, 1$

N N Y N N N N

$+ \quad - \quad + \quad +$

$-1 \qquad 1 \qquad \frac{9}{4}$

$=0 \qquad =0 \qquad =0$

≤ 0 $\quad$ " < " $\quad$?! whence comest this?

$\longrightarrow x \in \boxed{(-1, 1)} \rightarrow [-1, 1]$ Not

	$(-\infty, -1)$	$(-1, 1)$	$(1, \frac{9}{4})$	$(\frac{9}{4}, \infty)$
$(4x-9)^4$	+	+	+	+
$(x-1)^3$	−	−	+	+
$x+1$	−	+	+	+
	+	−	+	+

$(-1, 1)$

Check $x = -1, 1 \longrightarrow$ No. "< 0" not "≤ 0"

Y Y Y

N N Y N N N N

$+ \quad - \quad + \quad +$

$-1 \qquad 1 \qquad \frac{9}{4}$

$=0 \qquad =0 \qquad =0$

≤ 0, not < 0.

$\boxed{x \in [-1, 1] \cup \{\frac{9}{4}\}}$

19 Find all values of x for which the graph of f lies above the graph of g. (Enter your answer using interval notation.)

$$f(x) = \frac{1}{x}; \quad g(x) = \frac{1}{x-2}$$

Solve $f(x) > g(x)$

$$\frac{1}{x} > \frac{1}{x-2} \qquad \text{LCD: } x(x-2)$$

$$\frac{1}{x}\cdot\frac{(x-2)}{(x-2)} - \frac{1}{x-2}\cdot\frac{x}{x} > 0$$

$$\Rightarrow \frac{x-2-x}{x(x-2)} = \frac{-2}{x(x-2)} > 0$$

$-$ | 0 | $+$ | 2 | $-$

WANT >0 "+"

$\Rightarrow$ $\boxed{x \in (0,2)}$

Test-Value Version:

No NO | 0 | Yes | 2 | No NO

$(-\infty, 0)$ $\quad -1 \quad -1 = \frac{1}{-1} \overset{?}{>} \frac{1}{-1-2} = -\frac{1}{3}$ No

$(0,2)$ $\quad 1 \quad \frac{1}{1} = 1 \overset{?}{>} \frac{1}{1-2} = \frac{1}{-1} = -1$ Yes

$(2,\infty)$ $\quad 3 \quad \frac{1}{3} \overset{?}{>} \frac{1}{3-2} = \frac{1}{1}$ No

$\Rightarrow x \in (0,2)$

Sign Pattern:

$-$ No | 0 | $+$ | 2 No | $-$ $\quad > 0$

NO 0 Y 2 No

$\frac{-2}{x(x-2)}$ want > 0

$(-\infty,0)$ $\quad -1 \quad \frac{-2}{-1(-1-2)} = \frac{-2}{3} < 0$ No $-$

$(0,2)$ $\quad 1 \quad \frac{-2}{1(1-2)} = \frac{-2}{-1} > 0$ Yes $+$

$(2,\infty)$ $\quad 3 \quad \frac{-2}{3(3-2)} = \frac{-2}{3} < 0$ No $-$

$\boxed{x \in (0,2)}$