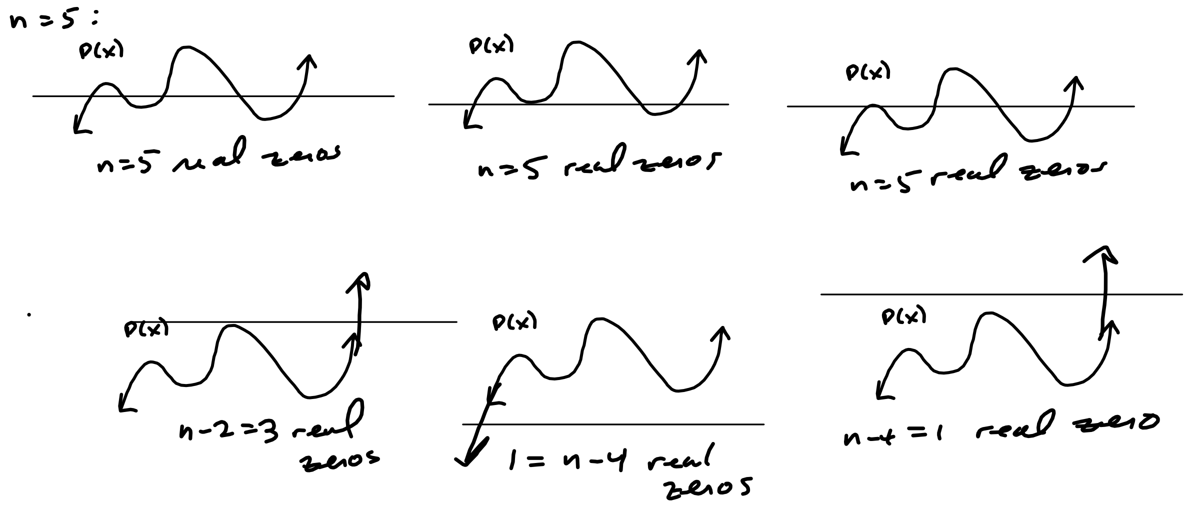
Section 3.5 - Complex Zeros and the Fundamental Theorem of Algebra

Recall: A polynomial $P(x)$, of degree n, has at most n - 1 extrema.

This means you can have at most n real zeros, and it requires n - 1 extrema and the maxima have to be above the x-axis and all the minima have to be below the x-axis.

But we know that this is a somewhat special situation, and you can have local maxima that are negative (below the x-axis) and we can have minima that are about the x-axis, i.e., positive.



The field of Complex Numbers is "algebraically closed." That means that every polynomial of degree n will have n complex zeros. They might all be real, as described, above, but they might not.

Fundamental Theorem
A polynomial of degree n has at least one complex zero.
By factor theorem:
$x=c$ is a zero means
$x-c$ is a factor. Earlier work showed how to use the Factor Theorem and Polynomial Division to "split off" a factor of $x-c$.

$P(x) = (x-c)Q(x)$, where $Q(x)$, the DEPRESSED POLYNOMIAL is of degree $n-1$

$x = c_1, c_2, c_3, \ldots, c_n$:

$$P(x) = (x-c_1)Q_1(x) = (x-c_1)(x-c_2)Q_2(x)$$

$$= \cdots = \underbrace{(x-c_1)(x-c_2)\cdots(x-c_n)}_{n \text{ factors}}$$

We count a root/zero of multiplicity m as m zeros

You will have:

$$(x-c_1)^{m_1}(x-c_2)^{m_2}\cdots(x-c_k)^{m_k}$$

where $m_1 + m_2 + \cdots + m_k = n =$ degree of $P(x)$.

A polynomial of degree n has either $n, n-2, n-4, \ldots, n-2j$ real zeros

This suggests the <u>Conjugate Pairs Theorem</u>

It says that complex, nonreal zeros occur in conjugate pairs, IF the coefficients of $P(x)$ are real.

$a+bi$ is a zero $\Longrightarrow$ $a-bi$ is a zero $(b \neq 0)$.

Suppose $ax^2+bx+c = P(x)$ has one non real zero

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \text{If } b^2-4ac < 0 \Longrightarrow$$

we get an i in front of the $\sqrt{}$

$$\frac{-b \pm i\sqrt{4ac-b^2}}{2a} = 2 \text{ conjugates!}$$

$$\frac{-b+i\sqrt{4ac-b^2}}{2a}, \quad \frac{-b-i\sqrt{4ac-b^2}}{2a}$$

So it has 2 non real zeros!

$P(x) = a(x-(c+di))(x-?) = P(x)$

we need $(x-(c+di))(x-?) = 0$

$x^2 - ?x + cx + dix + ?c + ?di$

$=0$ only if $? = c-di$

Why mess with all this stuff when we have computers and do everything digitally?

Because digital/decimal solutions always have some round-off to them. To defeat your graphing calculator, all I have to do is create a decimal that doesn't repeat until more digits than your calculator can handle.

While we *do* have Computer Algebra Systems that do *symbolic* manipulations that we're teaching, here, they can still let you down any time you go beyond their programmed abilities.

Build a fraction that's nastier than your calculator can handle.

$\frac{p}{q}$: Rational number is a decimal that terminates or that repeats.

$\underbrace{.23451278954}_{\text{Too many digits before repeat.}}\,23451278954\,23451278954\ldots = x$

$$10^{11}x = 23451278954.23451278954\,23451278954\ldots$$
$$x = .23451278954\,23451278954\ldots$$

$$10^{11}x - x = 23451278954$$

$$x(10^{11}-1) = 23451278954$$

$$x = \frac{23451278954}{10^{11}-1}$$ Exact, rational form.

99999999999

$x = 5.\overline{333}$

$10x = 53.\overline{3}$

$x = 5.\overline{3}$

$9x = 48.$

$\Rightarrow x = \frac{48}{9}$

Homework

1 The polynomial $P(x) = 5x^2(x-1)^3(x+9)$ has degree $\boxed{6}$. It has zeros 0, 1, and $\boxed{-9}$. The zero 0 has multiplicity $\boxed{2}$, and the zero 1 has multiplicity $\boxed{3}$.

2 (a) If a is a zero of the polynomial $P(x)$, then $\boxed{x-a}$ must be a factor of $P(x)$.

(b) If a is a zero of multiplicity m of the polynomial $P(x)$, then $\boxed{(x-a)^m}$ must be a factor of $P(x)$ when we factor P completely.

3 A polynomial of degree $n \geq 1$ has exactly $\boxed{n}$ zero(s), if a zero of multiplicity m is counted m times.

4 If the polynomial function P has real coefficients and if $\underbrace{a+bi}_{}$ ($z=?$) is a zero of P, then $\boxed{a-bi}$ ($\bar{z}=?$) is also a zero of P. So if $4+i$ is a zero of P, then $\boxed{4-i}$ is also a zero of P.

Assume $a, b \in \mathbb{R}$

$$(x-(a+bi))(x-(a-bi)) =$$

$$= x^2 - (a-bi)x - (a+bi)x + (a+bi)(a-bi)$$

$$= x^2 - ax \boxed{+bix} - ax \boxed{-bix} + a^2 \boxed{-abi} \boxed{+abi} - (bi)^2$$

$$= x^2 - 2ax + a^2 - b^2i^2 = x^2 - 2ax + a^2 + b^2$$

All coefficients are real.

True or False? If *False*, give a reason.

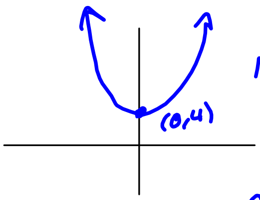
5 Let $P(x) = x^4 + 1$.

(a) The polynomial P has four complex zeros. True. Zeros Theorem (Pg 325)

(b) The polynomial P can be factored into linear factors with complex coefficients. True. By Zeros Theorem and the Factor Theorem.

$x=c$ makes $P(x)=0$, then $P(x)=(x-c)(Q(x))$ where $Q(x)$ is of degree $n-1$.

(c) Some of the zeros of P are real. False.



No x-intercepts
No real zeros.

Crash Course in Trigonometry & the Complex Plane (Optional)

From the Crash Course in Trigonometry

True or False? If *False*, give a reason.

5 Let $P(x) = x^4 + 1$. $\longrightarrow$ $x^4 = -1$

(a) The polynomial P has four complex zeros.

(b) The polynomial P can be factored into linear factors with complex coefficients.

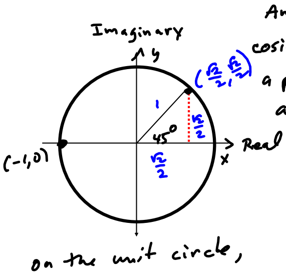
(c) Some of the zeros of P are real.

Fact: The Complex Numbers, the Complex PLANE, is precisely the number field we needed to guarantee that *every polynomial of degree* n *has exactly* n *zeros, if you count multiplicities as separate zeros.* It took a whole 2nd dimension to do it, hence the Complex *Plane*, versus the Real Number *Line*.

$a+bi = (a,b)$ $\qquad z = -1 = -1 + 0i = (-1, 0)$

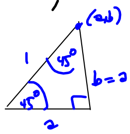
Angle is $180^\circ = \theta$ = Theta = "Thay–Tuh"

cosine gives us the x-coordinate for a point on the unit circle, based on the associated angle θ.



on the unit circle,

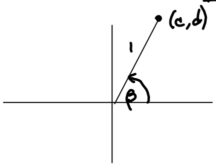
For a point with angle $\alpha = 45^\circ$, $\cos(45^\circ) = \frac{\sqrt2}{2}$



$a^2+b^2=1$
$a^2+a^2=1 \longrightarrow$
$2a^2=1 \longrightarrow$
$a^2=\frac12 \longrightarrow$
$a = \pm\sqrt{\frac12}$
$= \pm\frac{\sqrt1}{\sqrt2} = \pm\frac{1}{\sqrt2}\cdot\frac{\sqrt2}{\sqrt2}$
$= \frac{\sqrt2}{2}$

$\cos(45^\circ) = \frac{\sqrt2}{2}$
& Sine gives us the y-coordinate:
$\sin(\alpha) = \sin(45^\circ) = \frac{\sqrt2}{2}$

This means
I can write $-1 = -1 + 0i =$
I can do this for any angle! Any complex #.



Then $(c,d) = \cos\beta + i\sin\beta$

The principle 4th root of -1 is

$\sqrt[4]{-1} = \frac{\sqrt2}{2} + \frac{\sqrt2}{2}i$

$z = \cos(180^\circ) + i\sin(180^\circ)$

To get $\sqrt[4]{z} = \cos\left(\frac{180^\circ}{4}\right) + i\sin\left(\frac{180^\circ}{4}\right)$

$= \cos(45^\circ) + i\sin(45^\circ) = \frac{\sqrt2}{2} + \frac{\sqrt2}{2}i \longleftrightarrow \left(\frac{\sqrt2}{2}, \frac{\sqrt2}{2}\right)$

There are 3 more 4th roots. We get them by adding $\frac{360^\circ}{4}$ to each of the roots as we go:
This is our increment : $\frac{360^\circ}{4} = \frac{180^\circ}{2} = 90^\circ$

$\sqrt[4]{-1} = \frac{\sqrt2}{2} + \frac{\sqrt2}{2}i = \cos(45^\circ) + i\sin(45^\circ)$

Next: $= \cos(135^\circ) + i\sin(135^\circ)$

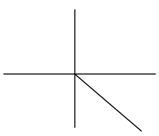
Next $= \cos(225^\circ) + i\sin(225^\circ)$

Finally $= \cos(315^\circ) + i\sin(315^\circ)$

360° $\qquad 2\pi$ Radians

$45^\circ = (45^\circ)\left(\frac{2\pi \text{ Radians}}{360^\circ}\right) = \frac{\pi}{4}$

$\frac{180^\circ}{\pi}$, $\frac{\pi}{180^\circ}$



$135^\circ = \frac{3\pi}{4}$
$225^\circ = \frac{5\pi}{4}$
$315^\circ = \frac{7\pi}{4}$

From the Crash Course in Trigonometry

Principal 4th root of $-1 = \cos(\pi) + i\sin(\pi)$

is $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \sqrt[4]{-1}$ & the other 3 are obtained

by adding the increment $\frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$:

$\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

$\frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$

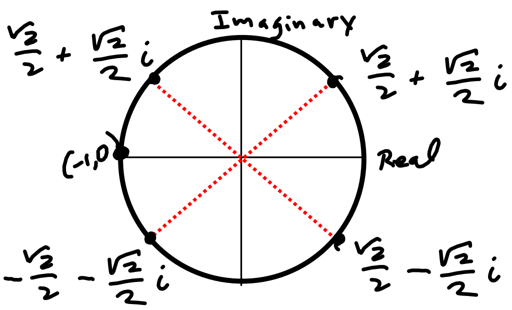
$\frac{5\pi}{4} + \frac{2\pi}{4} = \frac{7\pi}{4}$

$\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \sqrt[4]{-1}$

$\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$\sqrt[4]{z} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) =$



These are the 4 complex zeros of $P(x) = x^4 + 1$

It has no real zeros, so a graph is no help, other than to show you, visually, that it has no real solutions.

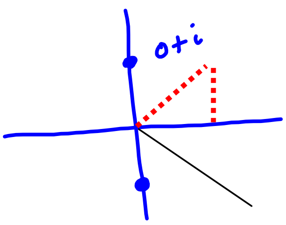
x^4+1

$= (x^2)^2 - (-1) = (x^2)^2 - i^2$

$u^2 - i^2 = (u-i)(u+i) \overset{SET}{=} 0$

$u = i \qquad u = -i$

$x^2 = i \qquad x^2 = -i$



reduces to finding the principal square root of i !

$\sqrt{x^2} = \sqrt{i}$

$x = \pm\sqrt{i}$

$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$\sqrt{i} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

$\sqrt[2]{\ }$ $\qquad \dfrac{\frac{\pi}{2}}{2}$

$\frac{2\pi}{2} = \pi = 180^\circ = \frac{360^\circ}{2}$

$45^\circ + 180^\circ = 225^\circ$

True or False? If *False*, give a reason.

6 Let $P(x) = x^3 + x$. $= x(x^2+1)$

(a) The polynomial P has three real zeros. FALSE. 0 is its only real zero. x^2+1 has zeros $\pm i$

$x^2 = -1$

$x = \pm\sqrt{-1} = \pm i$

(b) The polynomial P has at least one real zero.

TRUE x^3 End behavior

(c) The polynomial P can be factored into linear factors with real coefficients.

FALSE No. $P(x) = x(x-i)(x+i)$

Linear factors.

Nonreal coefficients

7 A polynomial P is given.

$P(x) = x^4 + 9x^2$

$x^2+9 = x^2 - (-9) = x^2 - (-1)(9) = x^2 - i^2(3)^2 = x^2 - (3i)^2$

(a) Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

$P(x) = x^2(x^2+9) \overset{\text{SET}}{=} 0 \rightarrow x = 0, \pm 3i$

$x^2(x-3i)(x+3i)$

$x^2 = -9$

$x = \pm\sqrt{-9} = \pm i\sqrt{9} = \pm 3i$

(b) Factor P completely.

A polynomial P is given.

8 $P(x) = x^5 + 64x^3$ $= x^3(x^2+64) = x^3(x^2-(8i)^2) = x^3(x-8i)(x+8i)$

(a) Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

$\Rightarrow$ 0's are $0, 0, 0, 8i, -8i$

(b) Factor P completely.

$x^3 = 0$ or $x^2 + 64 = 0$

$\Rightarrow x^2 = -64$

$x = \pm i\sqrt{64} = \pm 8i$

A polynomial P is given.

9 $P(x) = x^3 - 10x^2 + 41x$

(a) Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

$P(x) = x(x^2 - 10x + 41) \overset{SET}{=} 0 \rightarrow$

$x = 0$ or $x^2 - 10x + 41 = x^2 - 10x + 5^2 - 25 + 41 = (x-5)^2 + 16 \overset{SET}{=} 0 \Rightarrow$

$\boxed{x = 0, 5-4i, 5+4i}$ $(x-5)^2 = -16$

$x - 5 = \pm\sqrt{-16}$

$= \pm 4i \Rightarrow$

$x = 5 \pm 4i$

(b) Factor P completely.

$\boxed{P(x) = x(x-(5+4i))(x-(5-4i))}$

Consider the following expression.

10

$x^4 + 50x^2 + 625$ $= u^2 + 50u + 625$, where $u = x^2$. Then

Factor the given expression as much as possible.

$(u+25)^2 \overset{\text{set}}{=} 0 \rightarrow$

Perfect Square Trinomial

$u+25=0$

$x^2+2bx+b^2 = (x+b)^2$

$x^2+25=0$

$a^2x^2+2abx+b^2 = (ax+b)^2$

$x^2=-25$

$x=\pm 5i \rightarrow$

$(x^2+25)^2$

$=((x-5i)(x+5i))^2$

$= \boxed{(x-5i)^2(x+5i)^2}$

Let $P(x) = x^4 + 50x^2 + 625$

Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

$x = \pm 5i$, each of multiplicity $m=2$.

WebAssign expects: $5i, 5i, -5i, -5i$

11 **Let** $P(x) = x^3 + 216$

(a) Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

(b) Factor P completely.

$x^3 = -216$

$\sqrt[3]{x^3} = x = \sqrt[3]{-216} = \boxed{-6 = x \text{ is a zero}}$

Split off a factor of $x+6$

$$\begin{array}{r|rrrr} -6 & 1 & 0 & 0 & 216 \\ & & -6 & 36 & -216 \\ \hline & 1 & -6 & 36 & 0 \end{array}$$

sweet!

$P(x) = (x+6)(x^2-6x+36)$

$x^2-6x+36$ is our depressed polynomial

Solve $x^2-6x+36=0 \rightarrow$

$x^2-6x+3^2-9+36 = (x-3)^2+27 = 0 \rightarrow$

$(x-3)^2 = -27$

$x-3 = \pm\sqrt{-27} = 3\sqrt{3}i$

$x = 3 \pm 3\sqrt{3}i$ & $P(x) = (x+6)(x-(3+3\sqrt{3}i))(x-(3-3\sqrt{3}i))$

$x^3+216 = x^3+6^3 = (x+6)(x^2-6x+36)$

$x^2-6x+36 = x^2-6x+3^2-9+36$

$= (x-3)^2+27$

Find $\sqrt{-27}$ & write -27 as a square!

$= (x-3)^2-(-27)$

$= (x-3)^2-(3\sqrt{3}i)^2$

$$\begin{array}{r|l} 3 & 27 \\ 3 & 9 \\ & 3 \end{array}$$

$= ((x-3)-3\sqrt{3}i)((x-3)+3\sqrt{3}i)$

$= (x-3-3\sqrt{3}i)(x-3+3\sqrt{3}i)$

$\sqrt{27} = 3\sqrt{3}$

zeros: $x=-6, 3+3\sqrt{3}i, 3-3\sqrt{3}i$

$(x-(3+3\sqrt{3}i))(x-(3-3\sqrt{3}i))$

A polynomial P is given.

12 $P(x) = x^6 - 1$

(a) Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

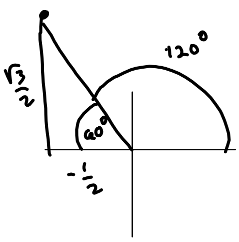
(b) Factor P completely.

$$x^6 - 1 = (x^3)^2 - 1^2 = (x^3 - 1)(x^3 + 1)$$
$$= (x-1)(x^2+x+1)(x+1)(x^2-x+1)$$
$$x^2+x+1 = x^2 + x + \left(\tfrac{1}{2}\right)^2 - \tfrac{1}{4} + \tfrac{4}{4}$$
$$= \left(x+\tfrac{1}{2}\right)^2 + \tfrac{3}{4} = 0$$
$$\Longrightarrow \left(x+\tfrac{1}{2}\right)^2 = -\tfrac{3}{4}$$
$$x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}} = \pm \tfrac{\sqrt{3}}{2} i$$
$$\Longrightarrow x = \frac{-1 \pm \sqrt{3} i}{2}$$
$$\vdots \quad x = \frac{1 \pm \sqrt{3} i}{2}$$

So, $P(x) = (x-1)(x+1)\left(x - \left(\frac{-1+\sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{-1-\sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{1+\sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{1-\sqrt{3}i}{2}\right)\right)$

$$-1, 1, \frac{-1 \pm \sqrt{3} i}{2}, \frac{1 \pm \sqrt{3} i}{2}$$

$$x^6 - 1 = (x^2)^3 - 1^3$$
$$= (x^2 - 1)((x^2)^2 + x^2 + 1)$$
$$= (x-1)(x+1)(x^4 + x^2 + 1)$$
$$u^2 + u + 1 = 0, \text{ where } u = x^2$$
$$u^2 + u + \left(\tfrac{1}{2}\right)^2 - \tfrac{1}{4} + \tfrac{4}{4}$$
$$= \left(u + \tfrac{1}{2}\right)^2 + \tfrac{3}{4} \overset{\text{SET}}{=} 0$$
$$= \left(u + \tfrac{1}{2}\right)^2 - \left(-\tfrac{3}{4}\right)$$
$$= \left(u + \tfrac{1}{2}\right)^2 - \left(\tfrac{\sqrt{3}}{2} i\right)^2$$
$$= \left(u + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right)\left(u + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right)$$
$$= \left(x^2 + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right)\left(x^2 + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right)$$

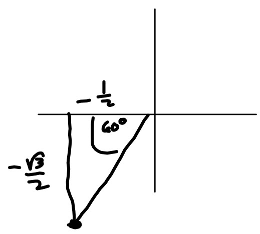


$$x^2 = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \quad \text{OR} \quad x^2 = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i$$

Need $\sqrt{-\frac{1}{2} + \frac{\sqrt{3}}{2} i}$?!

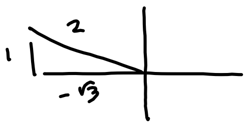
$$\cos\left(\tfrac{120^\circ}{2}\right) + i\sin(60^\circ)$$
$$= \cos(60^\circ) + i\sin(60^\circ)$$
$$= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \;!$$

As before.



Click Here for proof that $\cos(60^\circ) = \frac{1}{2}$ & $\sin(60^\circ) = \frac{\sqrt{3}}{2}$

$$\sqrt{z} = \cos\left(\tfrac{5\pi}{6}\right) + i\sin\left(\tfrac{5\pi}{6}\right)$$



$$= -\tfrac{\sqrt{3}}{2} + \tfrac{1}{2} i = x \;\&$$
$$+\tfrac{\sqrt{3}}{2} - \tfrac{1}{2} i = x$$

13

Let $P(x) = x^6 - 7x^3 - 8$

(a) Find all zeros of P, real and complex. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

(b) Factor P completely.

Let $x^3 = u$. Then

$u^2 - 7u - 8 = (u-8)(u+1) =$

$(x^3-8)(x^3+1)$

$x^3 = 8$

$x = \sqrt[3]{8} = 2$

$x^3 - 8 \div (x-2)$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -8 \\ & & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$ sweet!

$x^2+2x+4 = 0$

$\Rightarrow x^2+2x+1^2-1+4$

$= (x+1)^2 + 3 \overset{\text{SET}}{=} 0 \Rightarrow$

$(x+1)^2 = -3 \Rightarrow$

$x+1 = \pm i\sqrt{3}$

$\boxed{x = -1 \pm i\sqrt{3} \quad \& \quad 2}$

$x^3 = -1$

$x = \sqrt[3]{-1} = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & - \\ \hline & 1 & -1 & 1 & \end{array}$$ sweet!

$(x+1)(x^2-x+1)$

$x^2-x+1=0$

$\Rightarrow x^2 - x + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1$

$= \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} = 0 \Rightarrow$

$x = \frac{1 \pm \sqrt{3}i}{2}$

$-1, \frac{1 \pm \sqrt{3}i}{2}$

$(x+1)(x-2)(x-(-1+i\sqrt{3}))(x-(-1-i\sqrt{3}))\left(x-\left(\frac{1+\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1-\sqrt{3}i}{2}\right)\right)$

... is the linear factorization promised us!

20

Find a polynomial with integer coefficients that satisfies the given conditions.

↳ Integers are real

R has degree 4 and zeros $3 - 4i$ and 5, with 5 a zero of multiplicity 2.

$(x-(3-4i))\underbrace{(x-(3+4i))}_{\text{Conjugate Pairs Theorem}}(x-5)^2$

$(x-(3+4i))(x-(3-4i))$

$(x-3-4i)(x-3+4i) = x^2 - 3x + 4ix$
$\quad -3x \quad +9 \quad -12i$
$\quad -4ix \quad +16 \quad +12i$

$-16i^2 = 16$

$x^2 - 6x + 25$

$(x-5)^2 = x^2 - 10x + 25$

$P(x) = (x^2 - 10x + 25)(x^2 - 6x + 25)$

$= x^4 - 6x^3 + 25x^2$
$\quad -10x^3 + 60x^2 - 250x$
$\quad +25x^2 - 150x \quad +625$

$x^4 - 16x^3 + 110x^2 + 100x + 625$

$-400x$

Write Clearly!

Find all zeros (real and complex) of the polynomial. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

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$P(x) = x^4 - 6x^3 + 18x^2 - 54x + 81$

4, 2, 0 Positive

Descartes

$P(-x) = x^4 + 6x^3 + 18x^2 + 54x + 81$

No negative zeros

$$\begin{array}{r|rrrrr} 1 & 1 & -6 & 18 & -54 & 81 \\ & & 1 & -5 & 13 & \\ \hline & 1 & -5 & 13 & \text{Nope} & \end{array}$$

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & 18 & -54 & 81 \\ & & 3 & -9 & 27 & -81 \\ \hline 3 & 1 & -3 & 9 & -27 & 0 \text{ Sweet!} \\ & & 3 & 0 & 27 & \\ \hline 3 & 1 & 0 & 9 & 0 \text{ Sweet!} & \end{array}$$

$(x-3)^2(x^2+9) = (x-3)^2(x-3i)(x+3i)$

$x^2 - (-9) = x^2 - (3i)^2$

$x = 3, 3, -3i, 3i$