Section 3.4 - Real Zeros of Polynomials

By the time we're done with 3.4, we'll be ready for WP#3 #s 1 - 6.

Remainder Theorem $P(c)$ = remainder upon division of $P(x)$ by $x-c$.

Factor Theorem Remainder Theorem with $r=0$. zeros of $P(x)$ yield factors and conversely. $P(c)=0 \Rightarrow x-c$ is a factor.

Rational Zeros Theorem

Descartes' Rule of Signs

The Upper and Lower Bounds (on real zeros) Theorem

Using Algebra and Graphing Devices to Solve Polynomial Equations

1 If the polynomial function

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then the only numbers that could possibly be rational zeros of P are all of the form $\frac{p}{q}$, where p is a factor of ---Select--- and q is a factor of ---Select--- .

The possible rational zeros of $P(x) = 14x^3 + 5x^2 - 20x - 26$ are [ ] . (Enter your answers as a comma-separated list.)

This exercise is a copy-paste of the theorem?

RATIONAL ZEROS THEOREM

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form

$$\frac{p}{q}$$

$3x^5 - 7x^2 + 3$

$2x^4 + 9x^3 - 7x^2 + 5x - 7$

where p and q are integers and

p is a factor of the constant coefficient a_0

q is a factor of the leading coefficient a_n

$= 15x^2 - 11x - 14 = (3x+2)(5x-7)$

has zeros $x = -\frac{2}{3}, \frac{7}{5}$

$\frac{p}{q} = -\frac{2}{3}$: $p = 2$ is a factor of $a_0 = -14$; $q = 3$ is a factor of $a_n = a_2 = 15$

$\frac{p}{q} = \frac{7}{5}$: $p = 7$ is a factor of $a_0 = 14$; $q = 5$ is a factor of $a_n = a_2 = 15$

Examples:

$f(x) = 25x^5 + 10x^4 - 66x^3 - 80x^2 - 47x - 18$

has real zeros $x = -1, -1, 2$

$f(x) = (x-2)(x+1)^2(25x^2 - 15x + 24)$

2 is a factor of -18

-1 " " " " -18

$a_n = 2$

$a_0 = -174$

$$\begin{array}{r} 58 \\ 3\overline{)174} \\ -150 \\ \hline 24 \\ -24 \\ \hline 0 \end{array}$$

$f(x) = 2x^5 - 19x^4 + 166x^3 - 457x^2 + 482x - 174$

has zeros $x = 1, 1, \frac{3}{2}$ and

$x = 1$ is root of multiplicity $m = 2$.

factors over the reals as $(x-1)^2(x - \frac{3}{2})(2x^2 - 12x + 116)$

$\frac{p}{q} = \frac{3}{2}$ is a rational zero

Note: $p = 3$ is a factor of $a_0 = -174$

and $q = 2$ is a factor of $a_n = a_5 = 2$

The possible rational zeros of $P(x) = 14x^3 + 5x^2 - 20x - 26$ are

$\pm 1, \pm 2, \cancel{\pm \frac{2}{2}}, \pm \frac{2}{7}, \pm \frac{2}{14} = \pm \frac{1}{7}$

$\pm 13, \pm \frac{13}{2}, \pm \frac{13}{7}, \pm \frac{13}{14}, \pm \frac{26}{2} = \pm 13, \pm \frac{26}{7}, \cancel{\pm \frac{26}{14} = \pm \frac{13}{7}}$

$a_0 = -26$ $26 = 2 \cdot 13$

$a_n = 14$ $14 = 2 \cdot 7$

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{7}, \pm \frac{1}{14}$

2 Using Descartes's Rule of Signs, we can tell that the polynomial $P(x) = x^5 - 4x^4 + 7x^3 - x^2 + 9x - 5$ has, from smallest to largest, ______, ______, or ______ positive real zeros and ______ negative real zeros.

1 2 3 4 5

$f(-x) = -x^5 - 4x^4 - 7x^3 - x^2 - 9x - 5$

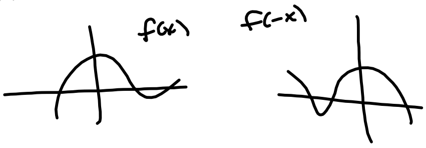
No sign changes

$\boxed{\text{Zero negative zeros}}$

$\boxed{5, 3, 1 \text{ possible positive zeros}}$

b/c $f(x)$ has 5 sign changes

For negative real zeros, we find the possible number of positive zeros for $f(-x)$.



3 If c is a real zero of the polynomial P, then all the other zeros of P are zeros of $P(x)/(x - c)$.

$\frac{P(x)}{x-c}$ is the depressed polynomial upon division by the factor $x-c$

THE DEPRESSED POLYNOMIAL of degree $n-1$. Repeat.... ... $n-2$, Repeat,.. ... $n-3$, until you're done.

TRUE!

$f(x) = x^3 - 2x^2 - 9x + 18$

$x^2(x-2) - 9(x-2)$

$= (x-2)(x^2-9) = (x-2)(x-3)(x+3)$

$x = 2 = c$ is a zero of $f(x)$.

& $\frac{f(x)}{x-2} = \frac{\cancel{(x-2)}(x-3)(x+3)}{\cancel{x-2}} = (x-3)(x+3) = x^2-9$

is the depressed polynomial.

Once you find a zero, $x=c$, split off the corresponding factor $x-c$, by division (or factoring) & the resulting quotient $\frac{f(x)}{x-c}$, the depressed polynomial, holds all the remaining factors/zeros of $f(x)$.

Procedure:

Solve $x^3 - 2x^2 - 9x + 18 = 0$

we found $x=2$ worked.

& $\frac{P(x)}{x-2} = x^2 - 9$ is what we're left with. That's 2^{nd} degree, we reduced the question to 2^{nd} degree from 3^{rd} degree

Divide by $x-2$:

$$
\begin{array}{r|rrrr}
2 & 1 & -2 & -9 & 18 \\
 & & 2 & 0 & -18 \\
\hline
 & 1 & 0 & -9 & 0
\end{array}
$$

Sweet!

Divide by $x-3$

$$
\begin{array}{r|rrr}
3 & 1 & 0 & -9 \\
 & & 3 & 9 \\
\hline
 & 1 & 3 & 0
\end{array}
$$

sweet!

This says $f(x) = (x-2)(x-3)(x+3)$

$(x+3)$

Never go back to the original $P(x)$! ONLY work with the depressed polynomial. Each zero you find reduces the degree of the problem by 1. Eventually, you run out of zeros!

4 If a is an upper bound for the real zeros of the polynomial P, then $-a$ is necessarily a lower bound for the real zeros of P.

FALSE!!!

Quick Counterexample:

$(x+1)(x-3) = x^2-2x-3$

has upper bound of 3

& lower bound of -1 for

its possible real zeros.

List all possible rational zeros given by the Rational Zeros Theorem (but don't check to see which actually are zeros). (Enter your answers as a comma-separated list.)

5

$U(x) = 12x^5 + 6x^3 - 3x - 8$

$a_n = 12 = 2 \cdot 2 \cdot 3$

$a_0 = -8 = -2 \cdot 2 \cdot 2$

$\frac{p}{q} = \frac{\text{factors of } a_0}{\text{factors of } a_n}$

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{6}, \pm \frac{1}{12},$

$\pm 2, \cancel{\pm \frac{2}{2}}, \pm \frac{2}{3}, \cancel{\pm \frac{2}{4}}, \cancel{\pm \frac{2}{6}}, \cancel{\pm \frac{2}{12}}$

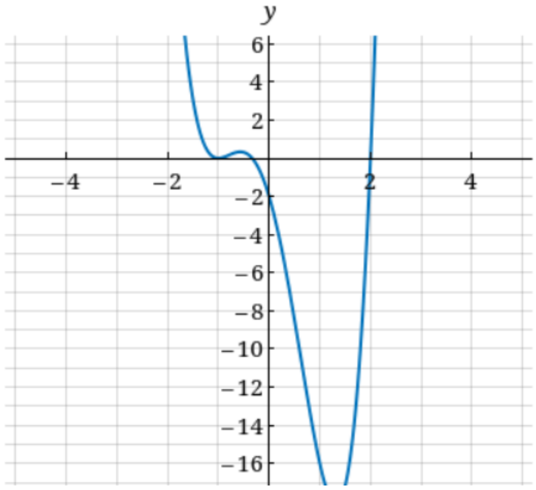
$\pm 4, \cancel{\pm \frac{4}{2}}, \pm \frac{4}{3}, \cancel{\pm \frac{4}{4}}, \cancel{\pm \frac{4}{6}}, \cancel{\pm \frac{4}{12}}$

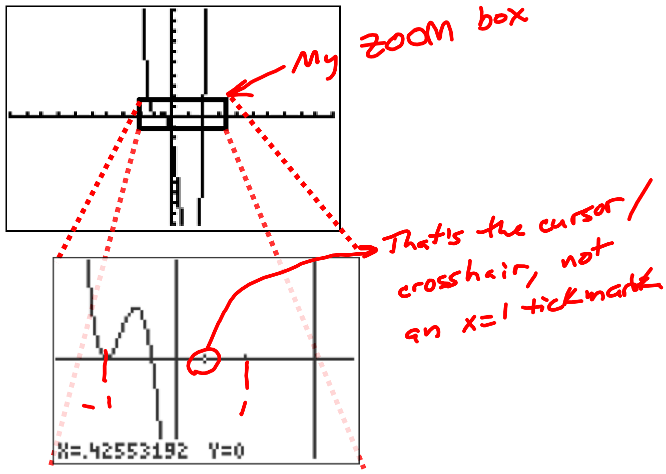
$\pm 8, \cancel{\pm \frac{8}{2}}, \pm \frac{8}{3}, \cancel{\pm \frac{8}{4}}, \cancel{\pm \frac{8}{6}}, \cancel{\pm \frac{8}{12}}$

6 A polynomial function P and its graph are given.

$P(x) = 3x^4 + x^3 - 9x^2 - 9x - 2$

[010,10]x[-10.10]





(a) List all possible rational zeros of P given by the Rational Zeros Theorem. (Enter your answers as a comma-separated list.)

(b) From the graph, determine which of the possible rational zeros actually turn out to be zeros. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

(a) $P(x) = 3x^4 + x^3 - 9x^2 - 9x - 2$

$a_0 = -2 \rightsquigarrow p\text{'s}$

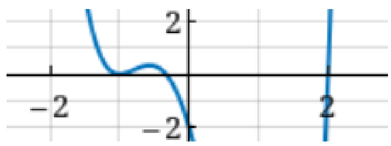
$a_n = 3 \rightsquigarrow q\text{'s}$

$\frac{p}{q} = \boxed{\pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}}$

$$\begin{array}{r|rrrrr}
-1 & 3 & 1 & -9 & -9 & -2 \\
 & & -3 & 2 & 7 & 2 \\
\hline
-1 & 3 & -2 & -7 & -2 & 0 \\
 & & -3 & 5 & 2 & \\
\hline
2 & 3 & -5 & -2 & 0 & \\
 & & 6 & 2 & & \\
\hline
 & 3 & 1 & 0 & &
\end{array}$$

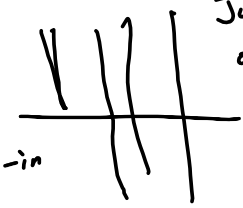
$\rightarrow 3x+1 = 0$ when $x = -\frac{1}{3}$

(b)



$\boxed{-1, -\frac{1}{3}, 2}$

Might be all you see. Just zero-in on the x-intercepts



Zoom-in

All the real zeros of the given polynomial are integers. Find the zeros. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

7

$P(x) = x^3 - 2x^2 - 13x - 10$

Descartes' ..

1 sign change 1 positive zero.

$P(-x) = -x^3 - 2x^2 + 13x - 10$

2 or 0 negative zeros

Write the polynomial in factored form.

$\pm 1, \pm 2, \pm 5, \pm 10$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -13 & -10 \\ & & 1 & -1 & -14 \\ \hline & 1 & -1 & -14 & \text{NO} \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -13 & -10 \\ & & -1 & 3 & 10 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

Sweet!

$$\begin{array}{r|rrr} -1 & 1 & -3 & -10 \\ & & -1 & 4 \\ \hline & 1 & -4 & \end{array}$$

Nope

This says

$P(x) = (x+1)(x^2 - 3x - 10)$

Depressed polynomial. It's quadratic, so we're done messing with guesses

Now $x^2 - 3x - 10 = (x-5)(x+2) \overset{\text{set}}{=} 0$

$\Rightarrow x \in \{-2, 5\}$

$\Rightarrow$ zeros of $P(x)$ are $-1, -2, 5$

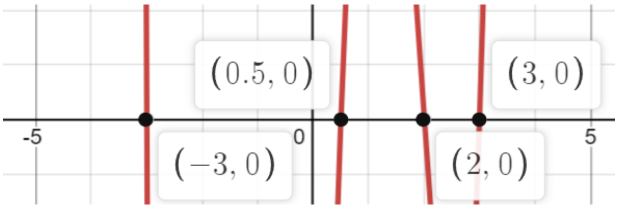
Write factored form:

$\boxed{P(x) = (x+1)(x+2)(x-5)}$

10 Find all rational zeros of the polynomial. (Enter your answers as a comma-separated list. Enter all answers including repetitions.)

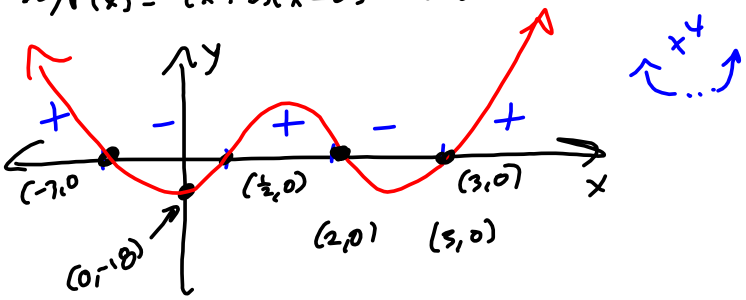
$$P(x) = 2x^4 - 5x^3 - 16x^2 + 45x - 18$$

Write the polynomial in factored form.



-3	2	-5	-16	45	-18	
		-6	33	-51	18	
2	2	-11	17	-6	0	Sweet!
		4	-14	6		
3	2	-7	3	0		sweet!
		6	-3			
	2	-1	0			
	x	c	r			

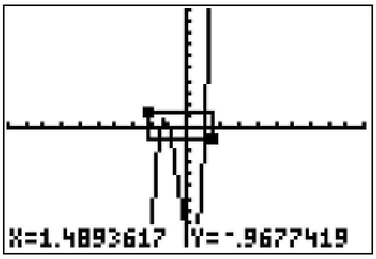
So, $P(x) = (x+3)(x-2)(x-3)(2x-1)$

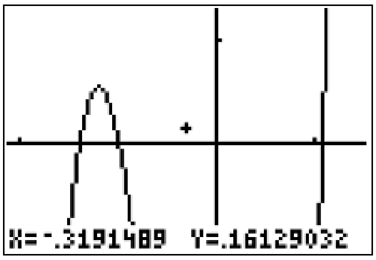


11

Find all the real zeros of the polynomial. Use the Quadratic Formula if necessary. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

$P(x) = 6x^3 + 8x^2 - 7x - 9$





$-1,$

$1+\varepsilon$

$-\frac{3}{2}$?

$$\begin{array}{r|rrrr} -1 & 6 & 8 & -7 & -9 \\ & & -6 & -2 & 9 \\ \hline & 6 & 2 & -9 & 0 \end{array}$$

Sweet!

Depressed: $6x^2+2x-9$

$(x+1)(6x^2+2x-9)$

$b^2-4ac = 2^2-4(6)(-9)$

$= 4+216 = 220$ Not a perfect square, ac method fails.

$2 \,|\, 220$, $2 \,|\, 110$, $5 \,|\, 55$, 11

$\sqrt{220} = 2\sqrt{55}$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

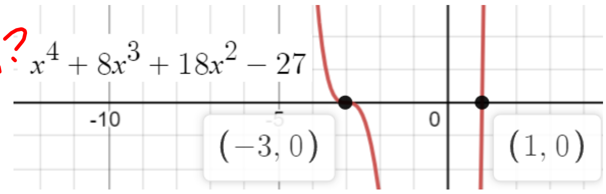
$$= \frac{-2 \pm 2\sqrt{55}}{2(6)} = \frac{-1 \pm \sqrt{55}}{6}$$

A polynomial P is given.

13 $P(x) = x^4 + 8x^3 + 18x^2 - 27$ → No x-term!

(a) Find all the real zeros of P. (Enter your answers as a comma-separated list. Enter all answers using the appropriate multiplicities.)

(b) Sketch the graph of P.



Where's the x term?

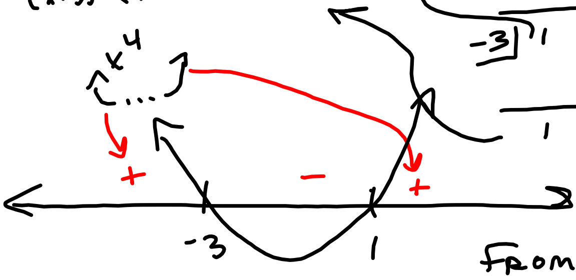
$$\begin{array}{r|rrrr} -3 & 1 & 8 & 18 & -27 \\ & & -3 & -15 & -9 \\ \hline & 1 & 5 & 3 & \text{Nope!} \end{array}$$

Need the placeholder

$$\begin{array}{r|rrrrr} -3 & 1 & 8 & 18 & 0 & -27 \\ & & -3 & -15 & -9 & 27 \\ \hline -3 & 1 & 5 & 3 & -9 & 0 \text{ Sweet!} \\ & & -3 & -6 & 9 & \\ \hline -3 & 1 & 2 & -3 & 0 & \text{Sweet!} \\ & & -3 & 3 & & \\ \hline & 1 & -1 & 0 \text{ sweet!} & & \end{array}$$

$(x+3)^2(x^2+2x-3)$

$(x+3)^3(x-1)^1$



FROM INO ONLY from 0's,
End Behavior & multiplicities (all odd)
CROSSES!

Show that the given values for a and b are lower and upper bounds for the real zeros of the polynomial.

15 $P(x) = 2x^3 + 7x^2 + 2x - 3$; $a = -4$, $b = 1$

for $a = -4$:

for $b = 1$:

Lower Bound:

$c \,\rfloor\; a_n \quad a_{n-1} \quad \ldots \quad a_2 \quad a_1 \quad a_0$

Alternating Signs.
"0" is the wildcard
(whatever it needs to be)

$$\begin{array}{r|rrrr} -4 & 2 & 7 & 2 & -3 \\ & & -8 & 4 & -24 \\ \hline & 2 & -1 & 6 & -27 \end{array}$$

Alternates $\Rightarrow$
$a = -4$ is L.B. on real zeros

Upper Bound

$c \,\rfloor\; a_n \quad a_{n-1} \quad a_{n-2} \quad \ldots \quad a_1 \quad a_0$

All positive #s
"0" gets the push.

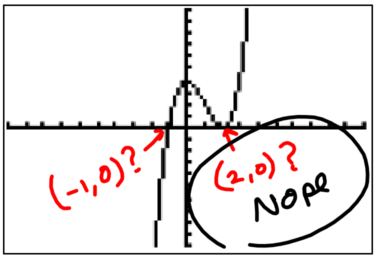
$b = 1$:

$$\begin{array}{r|rrrr} 1 & 2 & 7 & 2 & -3 \\ & & 2 & 9 & 11 \\ \hline & 2 & 9 & 11 & 8 \end{array}$$

All Positive $\Rightarrow$
$b = 1$ is U.B. on real zeros

17 Find integers that are upper and lower bounds for the real zeros of the polynomial. (Use the Upper and Lower Bounds Theorem. Be sure the lower bound is the largest possible lower bound and the upper bound is the smallest possible upper bound.)

$P(x) = x^3 - 3x^2 + 4$



0x place

-1	1	-3	0	4
		-1	4	-4
	1	-4	4	0
	+	−	+	−

Yes!

Make Double sure there's nothing bigger than -1 that's a negative integer. There aren't any!

0 can't work, b/c $-1 < 0$

& $P(-1) = 0$ makes it lower.

2	1	-3	0	4
		2	-2	-4
2	1	-1	-2	0
		2	2	0
	1	1	0	0

2 is U.B.!

1 -2 -

Nope.

$(x+1)(x^2-4x+4)$

$= (x+1)(x-2)^2$