3.1 Quadratic Functions and Models

We're "good" at completing the square to solve an equation. Probably less good at re-writing a quadratic function in "vertex form."

wp #1

$3x^2-30x+67=0$

$\Rightarrow x^2-10x+\frac{67}{3}=0$

$\Rightarrow x^2-10x+5^2=-\frac{67}{3}+25$

$\Rightarrow (x-5)^2=\frac{-67+75}{3}=\frac{8}{3}$

$\Rightarrow x-5=\pm\sqrt{\frac{8}{3}}=\pm\frac{2\sqrt{2}}{3}$

$\Rightarrow x=5\pm\frac{2\sqrt{2}}{3}$

wp #2

$f(x)=3x^2-30x+67$

$=3(x^2-10x\quad)+67$

$=3(x^2-10x+5^2)+67-3(25)$

$=3(x-5)^2-8=f(x)$

in vertex form $\Rightarrow$

$(h,k)=(5,-8)$

opens up.

New Way!

$f(x)=3x^2-30x+67 \Rightarrow$

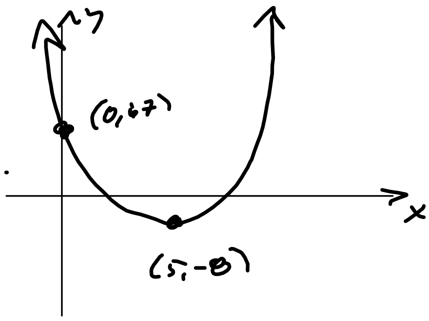
$\frac{f(x)}{3}=x^2-10x+\frac{67}{3}$

$=x^2-10x+5^2-25+\frac{67}{3}$

$=(x-5)^2-\frac{8}{3} \Rightarrow$

$f(x)=3(x-5)^2-8$

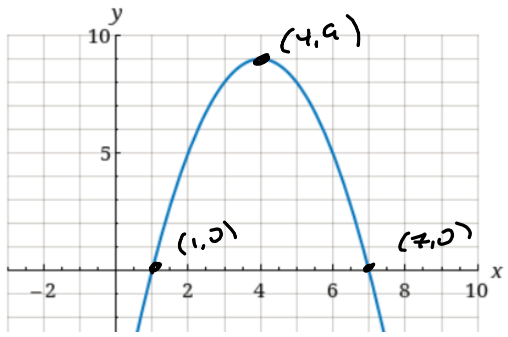
$(5,-8)$



#s 1 - 4 See Video

The graph of a quadratic function *f* is given.

5 $f(x) = -x^2 + 8x - 7$ → $(0,-7) = y\text{-int}$



(a) Find the coordinates of the vertex and the *x*- and ~~*y*-intercepts~~.

(b) Find the maximum or minimum value of *f*. Max is 9

(c) Find the domain and range of *f*. (Enter your answers using interval notation.)

$D = (-\infty, \infty)$, $R = (-\infty, 9]$

(a) y-int: $(0,-7)$
x-int: $(1,0)$, $(7,0)$
Vertex: $(4,9)$

A quadratic function *f* is given.

$f(x) = x^2 - 2x + 7$

7 (a) Express *f* in standard form.

(b) Find the vertex and *x*- and *y*-intercepts of *f*. (If an answer does not exist, enter DNE.)

(c) Sketch a graph of *f*.

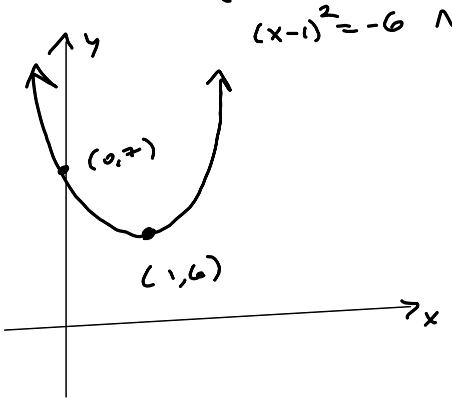
$f(x) = x^2 - 2x + 7 = x^2 - 2x + 1^2 - 1 + 7$

(a) $= \boxed{(x-1)^2 + 6 = f(x)}$ →

(b) $(h,k) = (1,6)$

$\boxed{y\text{-int}: (0,7)}$

x-int: $(x-1)^2 + 6 = 0$
$(x-1)^2 = -6$ No real sol'n!



10 A quadratic function f is given.

$$f(x) = -3x^2 + 6x - 1$$

(a) Express f in vertex form.

(b) Find the vertex and x- and y-intercepts of f. (If an answer does not exist, enter DNE.)

(c) Sketch a graph of f.

$f(x) = -3x^2 + 6x - 1$

$= -3(x^2 - 2x + 1^2) - 1 + 3(1)$

$= -3(x-1)^2 + 2 \quad (h,k) = (1,2)$

$f(x) = -3x^2 + 6x - 1 \Rightarrow$

$\frac{f(x)}{-3} = x^2 - 2x - \frac{1}{3} \to +\frac{1}{3}$. we're dividing by -3 not $+3$.

$= x^2 - 2x + 1^2 - \frac{1}{3} - 1 \to +\frac{1}{3}$

$= (x-1)^2 - \frac{4}{3} \to -\frac{2}{3}$

$f(x) = -3(x-1)^2 + 2$

Find the maximum or minimum value of the function.

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$$g(x) = 100x^2 - 1,400x$$

Two methods:

1. Compete the Square and find the vertex.

2. Use x-intercepts and symmetry to find h. Then find k via $k = f(h)$!

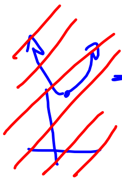
$g(x) = 100x^2 - 1400x$

$= 100(x^2 - 14x \quad)$

$= 100(x^2 - 14x + 7^2) - 100(49)$

$= 100(x-7)^2 - 4900$

$(h,k) = (7, -4900)$



$\Rightarrow \boxed{\text{Minimum Value is } -4900}$

$g(x) = 100x^2 - 1400x \Rightarrow$

$\frac{g(x)}{100} = x^2 - 14x$

$= x^2 - 14x + 7^2 - 49$

$= (x-7)^2 - 49 = \frac{g(x)}{100}$

$\Rightarrow 100((x-7)^2 - 49) = 100\left(\frac{g(x)}{100}\right)$

$\Rightarrow g(x) = 100(x-7)^2 - 4900$

$(7, -4900)$

Method 2: Use symmetry!

$g(x) \overset{SET}{=} 0 \Rightarrow$

$100(x-7)^2 - 4900 = 0$

$(x-7)^2 - 49 = 0$

$(x-7)^2 = 49$

$x - 7 = \pm 7$

$x = 7 \pm 7 \to x = 0, \; x = 14$

Easiest:

$100x^2 - 1400x = 100x(x - 14) \overset{SET}{=} 0 \Rightarrow$

$x = 0, 14$

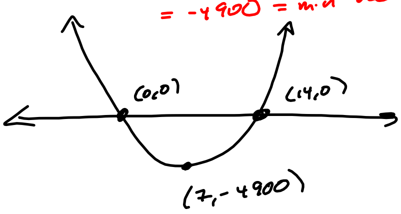
$\frac{14+0}{2} = 7$

$f(7) = 100(7)^2 - 1400(7)$

$= 4900 - 9800$

$= -4900 = $ min value

$\begin{array}{r} 2 \\ 14 \\ \times 7 \\ \hline 98 \end{array}$



Find the maximum or minimum value of the function.

15 $g(x) = 2x(x-4)+7$ $= 2x^2 +$ lower degree

Is this a maximum or minimum value? "Extrema"

I used Wolfram Alpha for this. Just clobbered it.

$32-32+7 = \cancel{x}+7 = \cancel{x}\ 7$

$2x^2-8x+7$

$= 2(x^2-4x+2^2)-8+7$

$= 2(x-2)^2-1$ min of -1 @ $x=2$

$2x^2-8x+7=0$

$a=2,\ b=-8,\ c=7$

$b^2-4ac = 64-4(2)(7)$

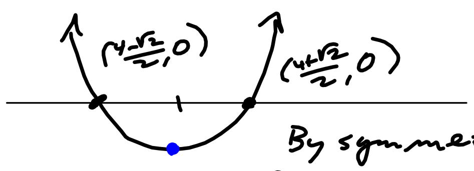
$= 64-56$

$= 8 \rightarrow \sqrt{8} = 2\sqrt{2}$

$2 \,|\, 8$, $2 \,|\, 4$, 2

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{+8 \pm \sqrt{8}}{2(2)} = \dfrac{8 \pm 2\sqrt{2}}{2(2)} = \dfrac{4 \pm \sqrt{2}}{2}$

$= \dfrac{4+\sqrt{2}}{2}$, $\dfrac{4-\sqrt{2}}{2}$



By symmetry of quadratic functions, the vertex lies halfway between:

$\dfrac{\frac{4-\sqrt{2}}{2} + \frac{4+\sqrt{2}}{2}}{2} = \dfrac{8}{4} = 2$

$g(x) = 2x(x-4)+7$

& $g(2) = 2(2)(2-4)+7$

$= 4(-2)+7$

$= -8+7$

$= -1$ = min value!

Click on me!

What I typed into Wolfram Alpha

Extrema of y=2*x*(x-4)+7

A graphing device is recommended.

16 A quadratic function f is given.

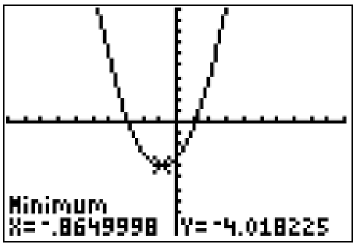
$$f(x) = x^2 + 1.73x - 3.27$$

$\frac{1.73}{2} = .865$ is exact

$(1.73)(.5)$

(a) Use a graphing device to find the maximum or minimum value of f, rounded to two decimal places.

(b) Find the exact maximum or minimum value of f, and compare it with your answer to part (a).



(a) $(-.86, -4.02)$

(b) $-.865 = h$

$k = \text{min value} = f(h)$

$= (-.865)^2 + 1.73(.865) - 3.27$

```
-.865²
             -.748225
(-.865)²+1.73*(-.
865)-3.27
            -4.018225
Ans▸Frac
            -4.018225
```

when in doubt about precision for part (b), try this:

$$\left(-\frac{865}{1000}\right)^2 + \left(\frac{173}{100}\right)\left(-\frac{865}{1000}\right) - \frac{327}{100}$$

$$\frac{865^2}{1000^2} - \frac{173(865)}{100(1000)} - \frac{327}{100}$$

$1000^2 = (10^3)^2 = 10^6$

$$\frac{865^2}{10^6} - \frac{173(865)}{10^5} \cdot \frac{10^1}{10^1} - \frac{327}{10^2} \cdot \frac{10^4}{10^4}$$

```
865²-173*865*10-
327*10^4
             -4018225
Ans/1000000
            -4.018225
Ans▸Frac
            -4.018225
█
```

$$= \frac{865^2 - 173(865)(10) - 327 \cdot 10^4}{10^6}$$

$$= \frac{-4018225}{10^6} = -4.018225$$

$= -160729/40000$

```
             -4018225
Ans/1000000
            -4.018225
Ans▸Frac
            -4.018225
-.865²-3.27
            -4.018225
█
```

clever.

ax^2+bx+c

$-\left(\frac{b}{2a}\right)^2 + c = f\left(-\frac{b}{2a}\right)$

$.73x - 3.27$

$.73x + (.865)^2 - (.865)^2 - 3.27$

$.865)^2 - (.865)^2 - 3.27$

ax^2+bx+c

$= a\left(x + \frac{b}{2a}\right)^2 + c - \left(\frac{b}{2a}\right)^2$

$\frac{4ac-b^2}{4a^2}$

$(h, k) = (-.865, -.865^2 - 3.27)$

$f(h) = -4.018225$

$k = f(h)$

$(h, k) = (h, f(h))$

```
             -4018225
Ans/1000000
            -4.018225
Ans▸Frac
            -4.018225
-.865²-3.27
            -4.018225
█
```

A graphing device is recommended.

17 A quadratic function f is given.

$$f(x) = 1 + x - \sqrt{7}x^2$$

(a) Use a graphing device to find the maximum or minimum value of f, rounded to two decimal places.

(b) Find the exact maximum or minimum value of f, and compare it with your answer to part (a).

Extrema of -sqrt(7)x^2+x+1

$$\max\{-\sqrt{7}\,x^2 + x + 1\} = 1 + \frac{1}{4\sqrt{7}} \text{ at } x = \frac{1}{2\sqrt{7}}$$

$$\frac{4\sqrt{7}}{4\sqrt{7}} + \frac{1}{4\sqrt{7}} = \frac{4\sqrt{7}}{4\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} + \frac{1}{4\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{28 + \sqrt{7}}{4 \cdot 7} = \frac{28 + \sqrt{7}}{28} = 1 + \frac{\sqrt{7}}{28}$$

$1 + x - \sqrt{7}\,x^2$

$= -\sqrt{7}\,x^2 + x + 1 \overset{\text{SET}}{=} 0$

$a = -\sqrt{7},\ b = 1,\ c = 1$

$b^2 - 4ac = 1^2 - 4(-\sqrt{7})(1)$

$= 1 + 4\sqrt{7}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{4\sqrt{7} + 1}}{2(-\sqrt{7})} = \frac{-1 \pm \sqrt{4\sqrt{7} + 1}}{-2\sqrt{7}}$$

Find midpoint of the 2 sol'ns

$$\frac{\dfrac{-1 + \sqrt{4\sqrt{7} + 1} + -1 - \sqrt{4\sqrt{7} + 1}}{-2\sqrt{7}}}{2}$$

$$= \frac{-2}{-4\sqrt{7}} = \frac{1}{2\sqrt{7}}$$

$$g\left(\frac{1}{2\sqrt{7}}\right) = -\sqrt{7}\left(\frac{1}{2\sqrt{7}}\right)^2 + \frac{1}{2\sqrt{7}} + 1$$

$$= -\sqrt{7}\left(\frac{1}{4 \cdot 7}\right) + \frac{1}{2\sqrt{7}} + 1$$

LCD: $28\sqrt{7}$

$$= \frac{-\sqrt{7}}{28} \cdot \frac{\sqrt{7}}{\sqrt{7}} + \frac{1}{2\sqrt{7}} \cdot \frac{14}{14} + \frac{28\sqrt{7}}{28\sqrt{7}}$$

$$= \frac{-7}{28\sqrt{7}} + \frac{+14}{28\sqrt{7}} + \frac{28\sqrt{7}}{28\sqrt{7}}$$

$$= \frac{7 + 28\sqrt{7}}{28\sqrt{7}} = \frac{1}{4\sqrt{7}} + 1 = \frac{\sqrt{7}}{28} + 1$$

$-\sqrt{7}\,x^2 + x + 1$

$-\sqrt{7}\left(x^2 - \frac{1}{\sqrt{7}}x \qquad \right) + 1$

$= -\sqrt{7}\left(x^2 - \frac{1}{\sqrt{7}}x + \left(\frac{1}{2\sqrt{7}}\right)^2\right) + \sqrt{7}\left(\frac{1}{4 \cdot 7}\right) + 1$

$= f(h)$!

$= \frac{\sqrt{7}}{28} + 1$

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Find a function f whose graph is a parabola that has the given vertex and that passes through the indicated point.

vertex $(6, -4)$; point$(7, 5)$

Quadratic Function in Standard or Vertex form.

$f(x) = a(x-h)^2 + k$

Vertex (h,k)

Given $(h,k) = (6,-4)$ and $(x,y) = (7,5)$ is on its graph.

$\Rightarrow$ we have $\boxed{f(x) = a(x-6)^2 - 4}$ &

$f(7) = 5$, i.e.,

$f(7) = a(7-6)^2 - 4$

$= a - 4 = 5$

$\Rightarrow \boxed{a = 9} \Rightarrow$

$$\boxed{f(x) = 9(x-6)^2 - 4}$$

A ball is thrown across a playing field from a height of 6 ft above the ground at an angle of 45° to the horizontal at the speed of 20 ft/s. (See the figure.)

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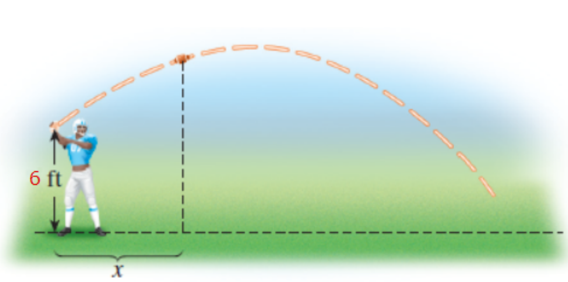

In trig, you'll be able to build this model, with *y* as a function of *x*, as the textbook authors have done. Usually we ease you into it, with *y* as a function of time *t*, in seconds, and the ball flies straight up and down (i.e., moves in 1 dimension).

It can be deduced from physical principles that the path of the ball is modeled by the function

$$y = -\frac{32}{(20)^2}x^2 + x + 6$$

where x is the distance (in feet) that the ball has traveled horizontally.

(a) Find the maximum height (in ft) attained by the ball. (Round your answer to three decimal places.)

(b) Find the horizontal distance (in ft) the ball has traveled when it hits the ground. (Round your answer to one decimal place.)

(a) Find 2 spots (x's) where $y=0$
Find the middle/average/midpoint of the two
Plug that in to find y.

(b) Use the "+" answer in part (a) to find when it hits the ground.

Solve $\frac{-32}{(20)^2}x^2 + x + 6 = 0$

$$\frac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} = h$$

Find $y(h) = \boxed{-\frac{32}{20^2}h^2 + h + 6} =$ Max Height

$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ is where it hits the ground

A Norman window has the shape of a rectangle surmounted by a semicircle, as shown in the figure below.

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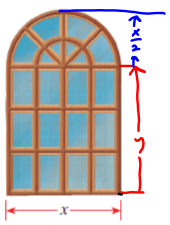

Let $A = A(x) =$ Area in ft^2 of the window as a function of $x =$ width of the window, in feet.

$A =$ Area $=$ Area of rectangle $+$ Area of $\frac{1}{2}$-circle

Let $y =$ height of rectangular part, in ft.

Then $A = xy +$ area of $\frac{1}{2}$ circle

A Norman window with perimeter 24 ft is to be constructed.

(a) Find a function that models the area (in ft^2) of the window, in terms of x. (Let x be the width, in ft, of the base of the window.)

(b) Find the dimensions (in ft) of the window that admits the greatest amount of light. (Round your answers to one decimal place.)

Maximize the AREA

$A = xy +$ Area of $\frac{1}{2}$-circle

Area of $\frac{1}{2}$-circle of radius $r = \frac{x}{2}$ is

$\frac{1}{2}$ Area of circle of radius $\frac{x}{2}$

$= \frac{1}{2}(\pi r^2) = \frac{1}{2}\left(\pi \left(\frac{x}{2}\right)^2\right) = \frac{\pi x^2}{8} \rightarrow$

$A = xy + \frac{\pi}{8}x^2$

Nugget: Perimeter is 24 ft.

$x + 2y + \underbrace{\text{Perimeter of } \tfrac{1}{2} \text{ circle of radius } r = \tfrac{x}{2}}$

$\frac{1}{2}$ perimeter of circle of radius $r = \frac{x}{2}$

$= \frac{1}{2}(2\pi r) = \frac{1}{2}\left(2\pi\left(\frac{x}{2}\right)\right)$

$= \frac{\pi x}{2}$

$\rightarrow$ $P = x + 2y + \frac{\pi x}{2} = 24$

This AUXILIARY equation gives y in terms of x:

Solve for y:

$2\left(x + 2y + \frac{\pi x}{2} = 24\right)$

$2x + 4y + \pi x = 48$

$4y = 48 - 2x - \pi x$

$y = \frac{48 - 2x - \pi x}{4} \rightarrow$

$A(x) = xy + \frac{\pi x^2}{8}$

$= x\left(\frac{48 - 2x - \pi x}{4}\right) + \frac{\pi x^2}{8}$

To find height of the window (in ft), we need

$h =$ height $=$ rect. part $+$ $\frac{1}{2}$-circle part

$= y + \frac{1}{2}x$

$= \frac{48 - 2x - \pi x}{4} + \frac{x}{2} \cdot \frac{2}{2}$

$= \frac{48 - 2x - \pi x + 2x}{4}$

$= \frac{48 - \pi x}{4}$

$\max\left\{\frac{1}{4}x(48 - 2x - \pi x) + \frac{\pi x^2}{8}\right\} \approx 40.3271390528321$ at $x \approx 6.72118984213869$