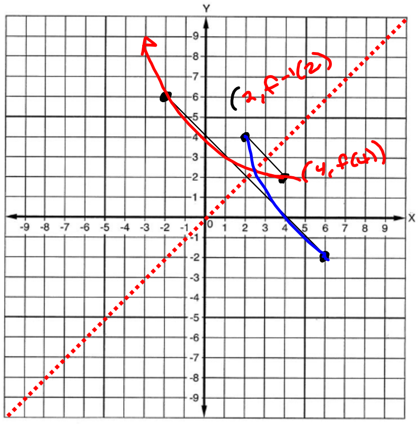
Section 2.8 - One-to-One Functions and their Inverses



Send y back to x

$f(x) = 7$
If you had an inverse function

$f^{-1}(x)$ such that

$f^{-1} \circ f = x$

$f(4) = 2$

I want $f^{-1}(f(4)) = f^{-1}(2) = 4$

$f(-2) = 6$
want $f^{-1}(6) = -2$

$y = f^{-1}$ is a reflection of $y = f$ about the line $y = x$.
That's what happens when you swap x & y

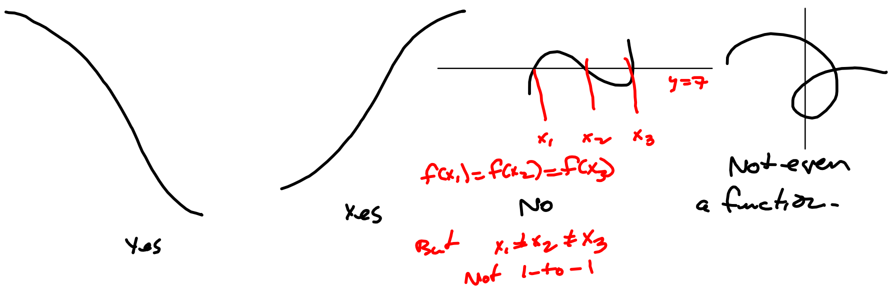
Now, we want f^{-1} to be a FUNCTION!

Suppose $f(2) = f(7) = 4$
We don't know how to define $f^{-1}(4)$.
$f^{-1}(4) = 2$? $f^{-1}(4) = 7$?
We have one y-value corresponding to 2 distinct x-values. Can't have that
That means $f(x)$ needs to be 1-to-1

That means we need $f(x)$ to be 1-to-1

The vertical line test for f^{-1} is The Horizontal Line Test for f.

1-to-1:



Definition
$f(x)$ is 1-to-1 if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
Equivalent: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (contrapositive)

$$A \Rightarrow B \iff \text{Not } B \Rightarrow \text{Not } A$$

Prove $f(x) = 3x + 1$ is 1-to-1, algebraically.

$\S$ $f(x_1) = f(x_2) \Rightarrow$

$3x_1 + 1 = 3x_2 + 1$

$3x_1 = 3x_2$

$x_1 = x_2$ $\blacksquare$

Find f^{-1}:

swap x & y, solve for y

$y = 3x+1$

$x = 3y+1 \Rightarrow$

$3y+1 = x \Rightarrow$

$3y = x-1 \Rightarrow$

$y = \boxed{\frac{x-1}{3} = f^{-1}(x)}$

Check: $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x+1)$

$= \frac{3x+1-1}{3} = \frac{3x}{3} = x$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-1}{3}\right) = 3\left(\frac{x-1}{3}\right) + 1$

$= x-1+1 = x$ ✓

But we have a lot of important functions that aren't 1-to-1, like x^2. We sure would like an inverse, and we already do! It's the Square Root Function!

$x^2 = 16$

$\sqrt{x^2} = \sqrt{16}$

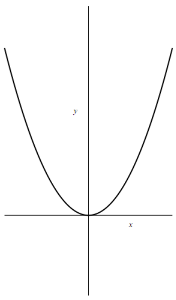
$|x| = 4$

$x = \pm 4$ No inverse <u>function</u> for

$f(x) = x^2$.

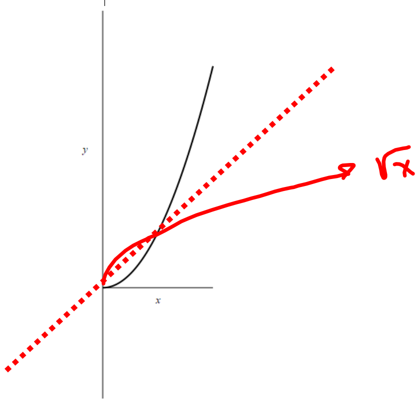
$f^{-1}(16) = 4$

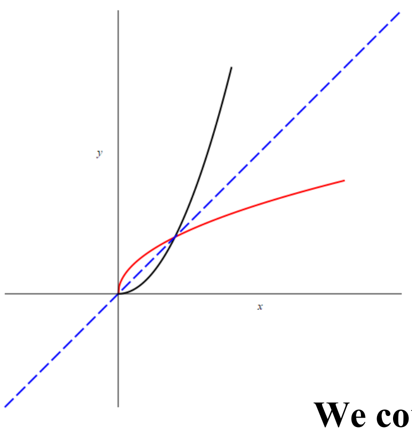
$f^{-1}(16) = -4$



It's 1-to-1 on either $(-\infty, 0]$ or $[0, \infty)$

Convention is $x \geq 0$, $[0, \infty)$

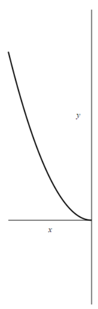


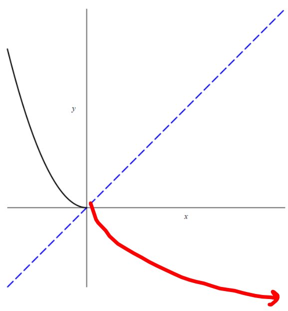


We could just as easily have restricted to nonpositive real numbers:

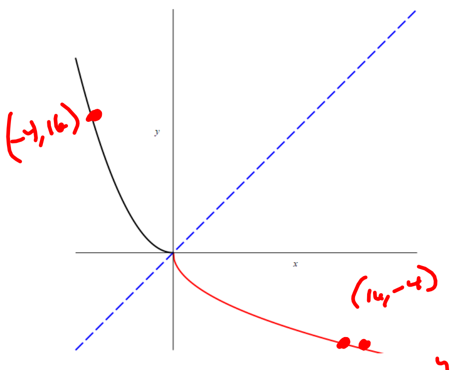
Addendum: **(You may safely scroll down to the Homework.**

We could just as easily have restricted *x* to nonpositive real numbers:





x^2 for $x \in (-\infty, 0)$



$y = -\sqrt{x}$

$f^{-1}(x) = -\sqrt{x}$

$f^{-1}(16) = -\sqrt{16} = -4$

Homework

1 A function f is one-to-one if different inputs produce [---Select---] outputs. You can tell from the graph that a function is one-to-one by using the [---Select---] Test.

horizontal different

2 (a) For a function to have an inverse, it must be [---Select---]. So which one of the following functions has an inverse?

1-to-1

- ○ $f(x) = x^2$
- ○ $g(x) = x^3$

(b) What is the inverse of the function that you chose in part (a)?

$y = x^3 \rightarrow x = y^3$ swap variables

$y^3 = x$ solve for y.

$\sqrt[3]{y^3} = \sqrt[3]{x}$

$y = \boxed{\sqrt[3]{x} = f^{-1}(x)}$

3 A function f has the following verbal description: "Multiply by 8, add 9, and then take the fifth power of the result."

(a) Write a verbal description for f^{-1}.

"Take the [---Select---] root, [---Select---] 9, then [---Select---] 8."

5th subtract divide by

(b) Find algebraic formulas that express f and f^{-1} in terms of the input x.

$f(x): (8x+9)^5$

$f^{-1}(x): \dfrac{\sqrt[5]{x} - 9}{8}$

$f^{-1}(f(x)) = f^{-1}((8x+9)^5) = \dfrac{\sqrt[5]{(8x+9)^5} - 9}{8}$

$= \dfrac{8x+9-9}{8} = \dfrac{8x}{8} = x$

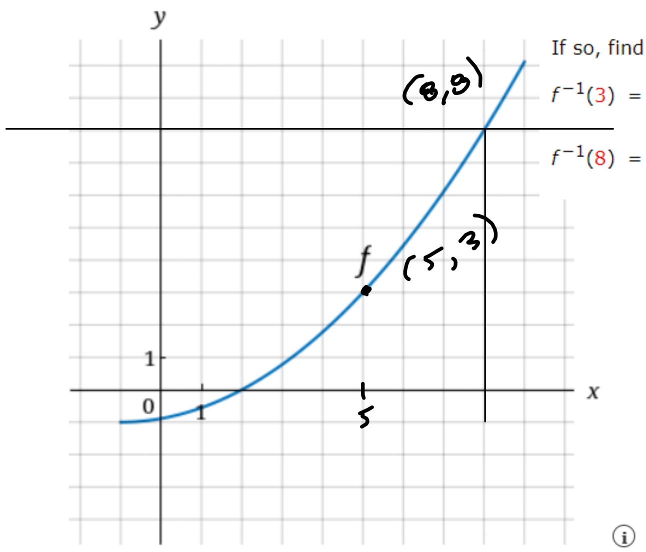
$f(f^{-1}(x)) = (8f^{-1} + 9)^5$

$= \left(8\left(\dfrac{\sqrt[5]{x} - 9}{8}\right) + 9\right)^5$

$= (\sqrt[5]{x} - 9 + 9)^5 = (\sqrt[5]{x})^5 = x$ ✓

A graph of a function f is given.

4



Does f have an inverse?

yes!

If so, find the following. (If an answer does not exist, enter DNE.)

$f^{-1}(3) = 5$

$f^{-1}(8) = 8$

5 If the point $(7, 1)$ is on the graph of the function f, then the point $(x, y) = (\boxed{1, 7})$ is on the graph of f^{-1}.

6 *True or False?*

(a) If f has an inverse, then $f^{-1}(x)$ is always the same as $\frac{1}{f(x)}$.

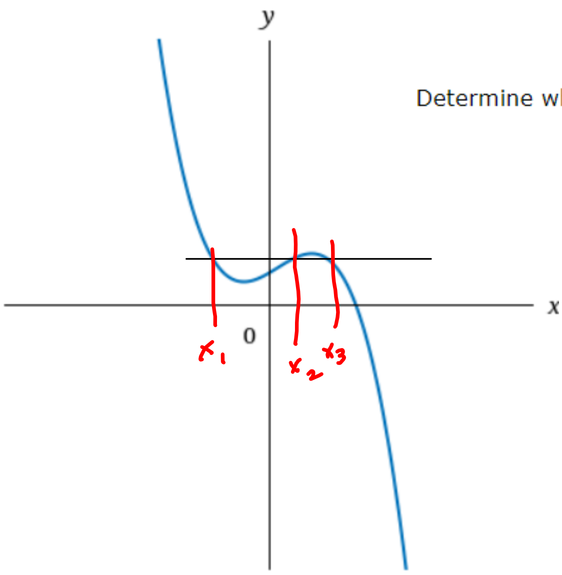
FALSE

→ is the arithMETic inverse of f(x). when we say $f^{-1}(x)$, we mean inverse with respect to function composition

(b) If f has an inverse, then $f^{-1}(f(x)) = x$.

True

7 A graph of a function f is given.



Determine whether f is one-to-one.

No

Horizontal line intersects the graph more than once.

$x_1 \neq x_2$, but $f(x_1) = f(x_2)$

9 Determine whether the function is one-to-one.

$f(x) = -3x + 4$

Suppose $f(x_1) = f(x_2) \Rightarrow$

$-3x_1 + 4 = -3x_2 + 4 \Rightarrow$

$-3x_1 = -3x_2 \Rightarrow$

$x_1 = x_2 \Rightarrow$ DONE ■

yes

11 Determine whether the function is one-to-one.

Not

$f(x) = |3x|$

Sp $|3x_1| = |3x_2|$

$\Rightarrow \quad 3x_1 = \pm 3x_2$

$x_1 = \pm x_2$

$f(-5) = |3(-5)| = |-15| = 15$

$f(5) = |3(5)| = |15| = 15$

$-5 \neq 5$, but $f(-5) = f(5) = 15 \Rightarrow$

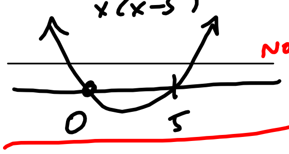
Not 1-to-1.

12 Determine whether the function is one-to-one.

$h(x) = x^2 - 5x$

Graphical

$x(x-5)$



All you need

The easy way to do the algebraic proof is to use the zeros!

$f(0) = f(5) = 0$

NOT 1-to-1.

Algebraic

$x_1^2 - 5x_1 = x_2^2 - 5x_2$

$x_1^2 - 5x_1 + \left(\frac{5}{2}\right)^2 = x_2^2 - 5x_2 + \left(\frac{5}{2}\right)^2$

$\left(x_1 - \frac{5}{2}\right)^2 = \left(x_2 - \frac{5}{2}\right)^2$ Take $\sqrt{\ }$:

$\Rightarrow \left|x_1 - \frac{5}{2}\right| = \left|x_2 - \frac{5}{2}\right|$

$\Rightarrow x_1 - \frac{5}{2} = \pm\left(x_2 - \frac{5}{2}\right)$

$x_2 - \frac{5}{2}$

$-x_2 + \frac{5}{2}$

$x_1 - \frac{5}{2} = x_2 - \frac{5}{2}$ OR $x_1 - \frac{5}{2} = -x_2 + \frac{5}{2}$

$x_1 = x_2$ $\boxed{x_1 = -x_2 + 5}$

Let $x_1 = 1$ (or ANYTHING $\neq \frac{5}{2}$)

Then $x_2 = 1$ OR $1 = -x_2 + 5$

$\Rightarrow \boxed{x_2 = 4}$

NOTE $f(1) = -4$

& $f(4) = 4^2 - 5(4) = 16 - 20 = -4$

So $x_1 \neq x_2$, but

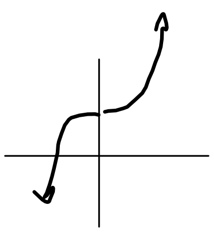
$f(x_1) = f(x_2)$

NOT 1-to-1

13 Determine whether the function is one-to-one.

$h(x) = x^3 + 6$

It's a cubic function moved up 6. It's 1-to-1.



Yeah.

$x_1^3 + 6 = x_2^3 + 6$

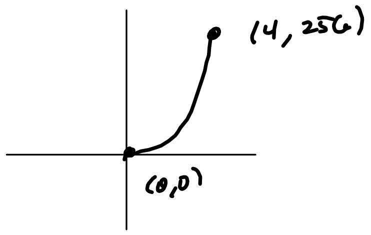
$x_1^3 = x_2^3$

$x_1 = x_2$

Determine whether the function is one-to-one.

15 $f(x) = x^4 + 3, \quad 0 \le x \le 4$

Aha!



Yes. 1-to-1, after the restriction

#s 17, 18 See #19

19 If $g(x) = x^2 + 8x$ with $x \ge -4$, find $\boxed{g^{-1}(48). = 4}$

Quick Way

Solve $x^2+8x = 48$

$\Rightarrow \quad x^2+8x-48=0$

$\Rightarrow \quad x^2+12x-4x-48$

$= x(x+12)-4(x+12)$

$= (x+12)(x-4) = 0$

$\Rightarrow \quad x \in \{-12, 4\}$

$-12 \notin [-4,\infty)$

Hard way (or if you had a lot of $g^{-1}(x)$'s to evaluate)

$y^2+8y = x$

$y^2+8y+4^2 = x+16$

$(y+4)^2 = x+16$

$y+4 = \pm\sqrt{x+16}$

$y = -4 \pm \sqrt{x+16}$ $\quad -4-\sqrt{x-16}$

$\boxed{y = -4 + \sqrt{x-16}} = g^{-1}(x)$ $\quad +$

b/c $x \ge -4$ in $\mathcal{D}(g)$ means $y \ge -4$ in $\mathcal{R}(g^{-1})$

$g^{-1}(48)$

$= -4+\sqrt{48-16}$ $\quad +$

$= -4+\sqrt{64}$

$= -4+8$ $\boxed{4 = g^{-1}(48)}$

A table of values for a one-to-one function is given. Find the indicated value.

20

x	1	2	3	4	5	6
$f(x)$	7	9	0	6	8	1

$f^{-1}(8) =$ 5

#s 21, 22 See #20

Use the Inverse Function Property to determine whether f and g are inverses of each other.

23 $f(x) = \frac{5}{x}$; $g(x) = \frac{5}{x}$

Yes

$f(g(x)) = \frac{5}{g(x)} = \frac{5}{\left(\frac{5}{x}\right)} = (5)\left(\frac{x}{5}\right) = x$ ✓

$g(f(x)) =$ SAME

Use the Inverse Function Property to determine whether f and g are inverses of each other.

27 $f(x) = \sqrt{64 - x^2}$, $0 \le x \le 8$;

$g(x) = \sqrt{64 - x^2}$, $0 \le x \le 8$

$f(g(x)) = \sqrt{64 - \left(\sqrt{64 - x^2}\right)^2} = \sqrt{64 - (64 - x^2)}$

$= \sqrt{64 - 64 + x^2} = \sqrt{x^2} = |x| \neq x$

But wait! The domain is $0 \le x \le 8$

& on $[0, 8]$, $x \ge 0 \rightarrow |x| = x$, so

yeah!

32

Find the inverse function of f. Check your answer by using the Inverse Function Property.

$$f(x) = \frac{3x+7}{x-5}$$

$$\frac{3y+7}{y-5} = x$$

$$3y+7 = x(y-5) = xy-5x$$

$$3y-xy = -5x-7$$

$$(3-x)y = -5x-7$$

$$y = \frac{-5x-7}{3-x} = \frac{-(5x+7)}{-(x-3)} = \boxed{\frac{5x+7}{x-3} = f^{-1}(x)}$$

Prove $f(x)$ is 1-to-1

$$f(x) = \frac{3x+7}{x-5}$$

$$\text{S } f(x_1) = f(x_2) \Rightarrow$$

$$\frac{3x_1+7}{x_1-5} = \frac{3x_2+7}{x_2-5} \Rightarrow$$

$$(3x_1+7)(x_2-5) = (3x_2+7)(x_1-5)$$

$$3x_1x_2 - 15x_1 + 7x_2 - 35 = 3x_2x_1 - 15x_2 + 7x_1 - 35$$

$= 3x_1x_2$

$$-15x_1 + 7x_2 = -15x_2 + 7x_1$$

$$-7x_1 \quad -7x_2 = -7x_2 - 7x_1$$

$$-22x_1 = -22x_2$$

$$x_1 = x_2 \quad \blacksquare$$

Find the inverse function of f. Check your answer by using the Inverse Function Property.

34

$f(x) = (x^3 - 8)^9$

$(y^3 - 8)^9 = x$

$y^3 - 8 = x^{\frac{1}{9}} = \sqrt[9]{x}$

$y^3 = \sqrt[9]{x} + 8$

$y = \sqrt[3]{\sqrt[9]{x} + 8}$