Section 2.1 Functions

A **function** f is a rule that assigns to each element x in a set **D**, called the **Domain,** exactly one element, called $f(x)$, in a set R, called the **range**.

In college algebra, the domain is typically the subset of the real numbers for which the rule is defined/real. In real life, domain is determined by the situation.

$$\mathbf{D} = \{x \mid f(x) \text{ is real}\}$$

$$\mathbf{R} = \{y \mid y = f(x) \text{ for some } x \text{ in } \mathbf{D}\}$$

Examples: $f(x) = 3x+2$, $f(x) = x^2$, $f(x) = \sqrt{x+2}$,

$f(x) = \frac{2}{x^2-1}$, $f(x) = \frac{7}{\sqrt{x^2-1}}$

In practice, $\mathcal{D}$ depends on 2 things:

(1) $\sqrt{\text{negative}}$ is bad ($\sqrt[2n]{\text{negative}}$)

(2) $\frac{\text{stuff}}{0}$ is bad

Faced with $\sqrt{\text{Junk}}$:

Need Junk ≥ 0. Solve it

$f(x) = \sqrt{2x-4}$ Need $2x-4 \geq 0$

$2x \geq 4$

$x \geq 2$

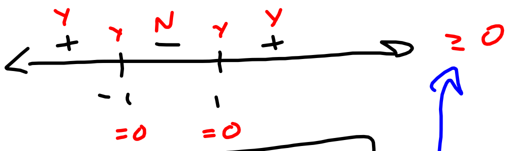
$\boxed{\mathcal{D} = [2,\infty)}$

$f(x) = \sqrt{x^2-1}$ $x^2-1 = (x+1)(x-1) \geq 0$

$(-\infty,-1)$: -2 $(-2)^2-1 = 3 > 0$ Y

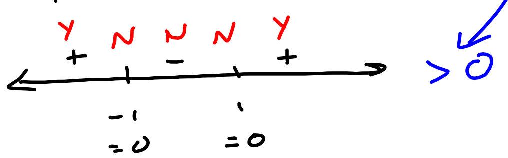
$(-1,1)$: 0 $0^2-1 = -1 < 0$ N

$(1,\infty)$: 2 $2^2-1 = 3 > 0$ Y



$\boxed{\mathcal{D} = (-\infty,-1] \cup [1,\infty)}$

$f(x) = \frac{7}{\sqrt{x^2-1}}$ Need $x^2-1 \geq 0$ AND $x^2-1 \neq 0$ $\}$ $x^2-1 > 0$



$= \boxed{(-\infty,-1) \cup (1,\infty) = \mathcal{D}}$

The **Range** is then all the y-values that are functions of x. All the heights achieved by the graph of f.

You really need an intuition about the graph.

1 If $f(x) = x^3 + 1$, then give the following.

$(-1)^3+1=0$

(a) the value of f at $x = -1$ is $f(-1) = 0$

$3^3+1 = 27+1$

(b) the value of f at $x = 3$ is $f(3) = 28$

(c) the net change in the value of f between $x = -1$ and $x = 3$ is $f(3) - f(-1) = 28$

Net change $= \Delta y = f(3) - f(-1) = 28 - 0 = 28$

2 For a function h, the set of all possible inputs is called the ---Select--- (Domain) of h, and the set of all possible outputs is called the ---Select--- (Range) of h.

3 (a) Which of the following functions have 9 in their domain? (Select all that apply.)

- ☐ $f(x) = x^2 - 3x$ — $\mathcal{D} = \mathbb{R}$
- ☐ $g(x) = \frac{x-9}{x}$ — $x \neq 0$ $\mathcal{D} = (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\} = \{x \in \mathbb{R} \mid x \neq 0\}$ $= \{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$
- ☐ $h(x) = \sqrt{x - 18}$ — $x - 18 \geq 0$, $x \geq 18$ & $9 < 18 \Rightarrow$ No!

(b) For the functions from part (a) that *do* have 9 in their domain, find the value of the function at 9. (If 9 is not in the domain, enter UNDEFINED.)

$f(x) = x^2 - 3x$ $\rightarrow f(9) = 9^2 - 3(9) = 81 - 27 = \boxed{54 = f(9)}$

$g(x) = \frac{x - 9}{x}$ $\rightarrow f(9) = \frac{9-9}{9} = \frac{0}{9} = \boxed{0 = g(9)}$

$h(x) = \sqrt{x - 18}$ undefined.

Keep it REAL.

4

Yes or No? If *No*, give a reason. Let f be a function.

Is it possible that $f(4) = 6$ and $f(5) = 6$?

- ○ Yes.
- ○ No. A function assigns each value of x in its domain to exactly one value of $f(x)$.
- ○ No. A function assigns each value of $f(x)$ in its range to exactly one value of x.
- ○ No. A function expecting a variable cannot be called with a constant argument.
- ○ No. There is no possible function operations that would yield 6 from 5.

5

Yes or No? If *No*, give a reason. Let f be a function.

Is it possible that $f(4) = 6$ and $f(4) = 8$?

- ○ Yes.
- ○ No. A function assigns each value of x in its domain exactly one value of $f(x)$.
- ○ No. A function assigns each value of $f(x)$ in its range exactly one value of x.
- ○ No. A function expecting a variable cannot be called with a constant argument.
- ○ No. There is no possible function operations that would yield 8 from 4.

Evaluate the function at the indicated values. (If an answer is undefined, enter UNDEFINED.)

6 $f(x) = x^2 + 5x$ $\mathcal{D} = \mathbb{R}$

$f(0) = 0^2 + 5(0) = \boxed{0 = f(0)}$

$f(3) = 3^2 + 5(3) = 9 + 15 = \boxed{24 = f(3)}$

$f(-3) = (-3)^2 + 5(-3) = 9 - 15 = \boxed{-6 = f(-3)}$

$\boxed{f(a) = a^2 + 5a}$

$f(-x) = (-x)^2 + 5(-x) = \boxed{x^2 - 5x = f(-x)}$

$f\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)^2 + 5\left(\frac{1}{a}\right) = \boxed{\frac{1}{a^2} + \frac{5}{a} = f\left(\frac{1}{a}\right)}$

Evaluate the function at the indicated values. (If an answer is undefined, enter UNDEFINED.)

7 $h(t) = t + \frac{3}{t}$

$h(-1) = (-1) + \frac{3}{-1} = -1 - 3 = \boxed{-4 = h(-1)}$

$h(4) = 4 + \frac{3}{4} = \frac{4}{1} \cdot \frac{4}{4} + \frac{3}{4} = \frac{16+3}{4} = \boxed{\frac{19}{4} = h(4)}$

$h\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{3}{\frac{1}{2}} = \frac{1}{2} + \frac{3}{1} \cdot \frac{2}{1} = \frac{1}{2} + \frac{6}{1} \cdot \frac{2}{2} = \boxed{\frac{13}{2} = h\left(\frac{1}{2}\right)}$

$\boxed{h(x-1) = (x-1) + \frac{3}{x-1}}$

$h\left(\frac{1}{x}\right) = \frac{1}{x} + \frac{3}{\left(\frac{1}{x}\right)} = \frac{1}{x} + \frac{3}{1} \cdot \frac{x}{1} = \boxed{\frac{1}{x} + 3x = h\left(\frac{1}{x}\right)}$

Evaluate the piecewise-defined function at the indicated values.

8

$$f(x) = \begin{cases} x^2 + 6x & \text{if } x \le -1 \\ x & \text{if } -1 < x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$$

$5\overline{)95}$ 19

$f(-3) = (-3)^2 + 6(-3) = 9 - 18 = \boxed{-9 = f(-3)}$

$f\left(-\frac{5}{4}\right) = \left(-\frac{5}{4}\right)^2 + 6\left(-\frac{5}{4}\right) = \frac{25}{16} - \left(\frac{30}{4}\right)\left(\frac{4}{4}\right) = \frac{25 - 120}{16} = \boxed{-\frac{95}{16} = f\left(-\frac{5}{4}\right)}$

$f(-1) = (-1)^2 + 6(-1) = 1 - 6 = \boxed{-5 = f(-1)}$

$f(0) = 0$

$f(45) = -1$

Use the function to evaluate the indicated expressions and simplify.

9

$$f(x) = 3x^2 + 7$$

$f(x + 2) = 3(x+2)^2 + 7 = 3(x^2 + 4x + 4) + 7 = 3x^2 + 12x + 12 + 7$
$= \boxed{3x^2 + 12x + 19 = f(x+2)}$

$f(x) + f(2) = (3x^2 + 7) + (3(2)^2 + 7) = 3x^2 + 7 + 3(4) + 7$
$= \boxed{3x^2 + 26 = f(x) + f(2)}$

NOTE: $f(x+2) \neq f(x) + f(2)$

Use the function to evaluate the indicated expressions and simplify.

10 $f(x) = 10x - 15$

$f\left(\frac{x}{5}\right) = 10\left(\frac{x}{5}\right) - 15 = \boxed{2x - 15 = f\left(\frac{x}{5}\right)}$

$\frac{f(x)}{5} = \frac{10x - 15}{5} = \boxed{2x - 3 = \frac{f(x)}{5}}$

Find the net change in the value of the function between the given inputs.

11 $h(t) = t^2 + 3$; from -4 to 7

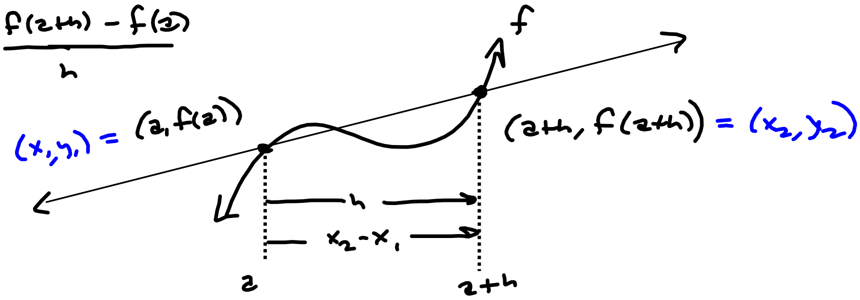
$h(7) - h(-4) = (7^2 + 3) - ((-4)^2 + 3)$

$= 49 + 3 - 16 - 3 = \boxed{33 = h(7) - h(-4)}$

Difference Quotient: $\frac{f(a+h) - f(a)}{h}$

$\frac{f(a+h) - f(a)}{h} = \frac{y_2 - y_1}{x_2 - x_1}$

= Average Slope of $f(x)$ on/over $[a, a+h]$



Find $f(a)$, $f(a + h)$, and the difference quotient $\frac{f(a + h) - f(a)}{h}$, where $h \neq 0$.

$f(x) = 6$ is horizontal line $y = f(x) = 6$

12 $f(a) = 6$

$f(a + h) = 6$

$\frac{f(a + h) - f(a)}{h} = \frac{6 - 6}{h} = \frac{0}{h} = 0$

Find $f(a)$, $f(a+h)$, and the difference quotient $\frac{f(a+h)-f(a)}{h}$, where $h \neq 0$.

13

$$f(x) = \frac{x}{x+7}$$

$$\boxed{f(a) = \frac{a}{a+7}}$$

$$f(a+h) = \frac{a+h}{(a+h)+7}$$

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{a+h}{a+h+7} - \frac{a}{a+7}}{h} = \frac{1}{h}\left[\frac{a+h}{a+h+7} - \frac{a}{a+7}\right]$$

$$= \frac{1}{h}\left[\left(\frac{a+h}{a+h+7}\right)\left(\frac{a+7}{a+7}\right) - \left(\frac{a}{a+7}\right)\left(\frac{a+h+7}{a+h+7}\right)\right]$$

$$LCD = (a+h+7)(a+7)$$

$$= \frac{1}{h}\left[\frac{(a+h)(a+7) - a(a+h+7)}{LCD}\right] = \frac{1}{h}$$

$$= \frac{1}{h}\left[\frac{a^2+7a+ah+7h-(a^2+ah+7a)}{LCD}\right]$$

$$= \frac{1}{h}\left[\frac{a^2+7a+\boxed{ah}+7h-a^2-\boxed{ah}-7a}{LCD}\right] =$$

$$= \frac{1}{\cancel{h}}\left[\frac{7\cancel{h}}{LCD}\right] = \boxed{\frac{7}{(a+7)(a+h+7)} = \frac{f(a+h)-f(a)}{h}} \quad h \neq 0$$

$$\xrightarrow{h \to 0} \frac{7}{(a+7)(a+7)} = \frac{7}{(a+7)^2}$$

CALCULUS I

Find $f(a)$, $f(a+h)$, and the difference quotient $\frac{f(a+h)-f(a)}{h}$, where $h \neq 0$.

14

$f(x) = 4 - 2x + 3x^2 = 3x^2 - 2x + 4$

$\boxed{f(a) = 3a^2 - 2a + 4}$

$(a+h)^2 = a^2 + 2ah + h^2$

$f(a+h) = 3(a+h)^2 - 2(a+h) + 4 = 3(a^2+2ah+h^2) - 2a - 2h + 4$

$= 3a^2 + 6ah + 3h^2 - 2a - 2h + 4 = \cancel{3a^2 + 6ah + 3h^2 - 2a - 2h + 4}$

$\frac{f(a+h)-f(a)}{h} = \frac{3a^2 + 6ah + 3h^2 - 2a - 2h + 4 - (3a^2 - 2a + 4)}{h}$

$= \frac{3a^2 + 6ah + 3h^2 - 2a - 2h + 4 - 3a^2 + 2a - 4}{h}$

$= \frac{6ah - 2h}{h} = \frac{h(6a-2)}{h} = \boxed{6a - 2 = \frac{f(a+h)-f(a)}{h} \quad (h \neq 0)}$

Find $f(a)$, $f(a+h)$, and the difference quotient $\frac{f(a+h)-f(a)}{h}$, where $h \neq 0$.

15

$f(x) = 2x^3$

$\boxed{f(a) = 2a^3} = ($

$f(a+h) = 2(a+h)^3 = 2(a+h)(a+h)(a+h) = 2(a+h)(a^2 + 2ah + h^2)$

$= 2(a^3 + 2a^2h + ah^2 + a^2h + 2ah^2 + h^3)$

$= 2(a^3 + 3a^2h + 3ah^2 + h^3)$

$= \boxed{2a^3 + 6a^2h + 6ah^2 + 2h^3 = f(a+h)}$

$\frac{f(a+h)-f(a)}{h} =$

1

1 1

1 2 1 → $a^2 + 2ah + h^2$

1 3 3 1 → $a^3 + 3a^2h + 3ah^2 + h^3$

1 4 6 4 1 → $a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4$

$\frac{f(a+h)-f(a)}{h} = \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 - 2a^3}{h} = \frac{6a^2h + 6ah^2 + 2h^3}{h}$

$= \frac{h(6a^2 + 6ah + 2h^2)}{h} = \boxed{6a^2 + 6ah + 2h^2 = \frac{f(a+h)-f(a)}{h} \quad h \neq 0}$ We want

Calc: $h \to 0 \Rightarrow 6a^2 = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Calc wants

Find the domain and range of the function. (Enter your answers using interval notation.)

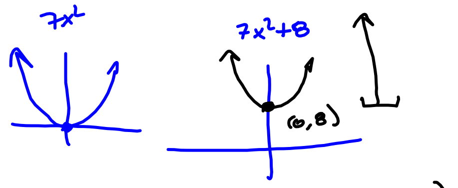
16 $f(x) = 7x^2 + 8$ $f(0)=8$

$x^2 \geq 0$

$7x^2 \geq 0$

$7x^2+8 \geq 8$

$\mathcal{D} = [8,\infty)$



$(-\infty \to \infty)$

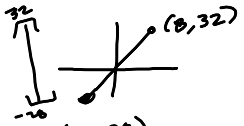
$(-\infty,\infty) = \mathcal{D} = \mathbb{R}$

Find the domain and range of the function. (Enter your answers using interval notation.)

17 $f(x) = 4x, \quad -7 \leq x \leq 8$

domain $= [-7, 8]$

range $= [-28, 32]$



$[-7, 8]$

Find the domain of the function. (Enter your answer using interval notation.)

18 $f(x) = \dfrac{x+8}{x^2-9}$

Need $x^2-9 \neq 0$

$(x+3)(x-3) \neq 0$

$x+3 \neq 0$ AND $x-3 \neq 0$

Recall

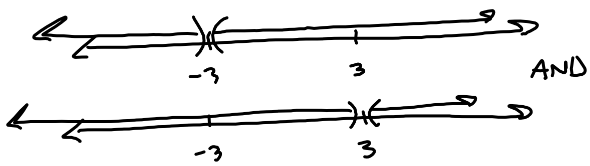
$x^2-9=0$

$(x+3)(x-3)=0$

$x+3=0$ OR $x-3=0$

NOT (A OR B) = NOT A AND NOT B

$x \neq -3$ and $x \neq 3$



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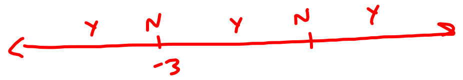


$= (-\infty,-3) \cup (-3,3) \cup (3,\infty)$

$= \mathbb{R} \setminus \{\pm 3\}$

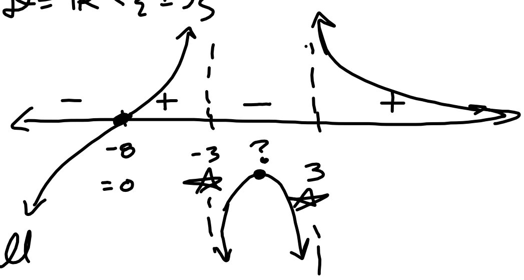
Range not Included.

$\dfrac{x+8}{(x+3)(x-3)} = \dfrac{x+8}{x^2-9}$

$x+8=0$

$x=-8$

$\mathcal{D} = \mathbb{R} \setminus \{\pm 3\}$



$\dfrac{x+8}{x^2-9}$ $x \to \text{BIG}$

$\dfrac{\text{BIG}}{\text{BIG}^2} = \dfrac{1}{\text{BIG}} = \text{Small}$

$x \to \infty, \ f(x) \to 0$

$\mathcal{R} = (-\infty,\infty)$ by graphing technique, Chapter 3

Find the domain of the function. (Enter your answer using interval notation.)

19 $f(x) = \dfrac{x^4}{x^2 + x - 12}$

$x^2 + x - 12 \neq 0$

$(x+4)(x-3) \neq 0$

$x+4 \neq 0$ and $x-3 \neq 0$

$x \neq -4$ and $x \neq 3$

$x \neq -4$ or 3

$\mathcal{D} = \mathbb{R} \setminus \{-4, 3\}$

$\boxed{\mathcal{D} = (-\infty, -4) \cup (-4, 3) \cup (3, \infty)}$

Find the domain of the function. (Enter your answer using interval notation.)

20 $g(x) = \sqrt{8 - 7x}$

Need $8 - 7x \geq 0$

$-7x \geq -8$

Key move

$x \leq \frac{-8}{-7} = \frac{8}{7}$

$\boxed{\mathcal{D} = (-\infty, \frac{8}{7}]}$

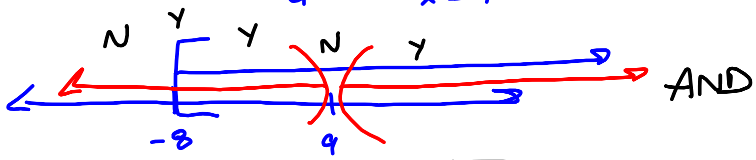
Find the domain of the function. (Enter your answer using interval notation.)

21 $g(x) = \dfrac{\sqrt{8+x}}{9-x}$

Need $8+x \geq 0$ and $9-x \neq 0$

$x \geq -8$ $\quad$ $-x \neq -9$

$x \neq 9$



$= \boxed{[-8,9) \cup (9,\infty) = \mathcal{D}}$

Find the domain of the function. (Enter your answer using interval notation.)

22 $g(x) = \dfrac{\sqrt{x}}{7x^2 + 6x - 1}$

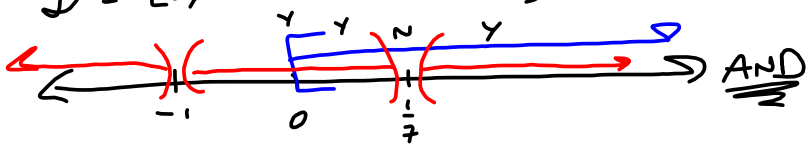
Need $x \geq 0$ AND $7x^2+6x-1 \neq 0$

$(7x-1)(x+1)$

$x \neq \left(\frac{1}{7} \text{ OR } -1\right)$

$x \neq \frac{1}{7}$ AND $x \neq -1$

$\mathcal{D} = [0,\infty) \cap \mathbb{R} \setminus \{-1, \frac{1}{7}\}$



$= \boxed{[0, \frac{1}{7}) \cup (\frac{1}{7}, \infty) = \mathcal{D}}$

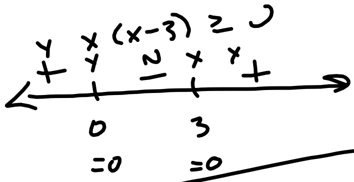
Find the domain of the function. (Enter your answer using interval notation.)

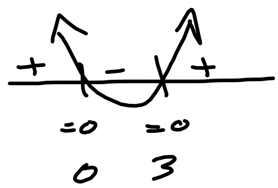
23 $g(x) = \sqrt[4]{x^2 - 3x}$

even-index root
Need radicand ≥ 0

Need $x^2 - 3x \geq 0$

$x(x-3) \geq 0$





$= \boxed{(-\infty, 0] \cup [3, \infty) = \mathcal{D}}$

Find the domain of the function. (Enter your answer using interval notation.)

24 $g(x) = \sqrt{x^2 - 2x - 15}$

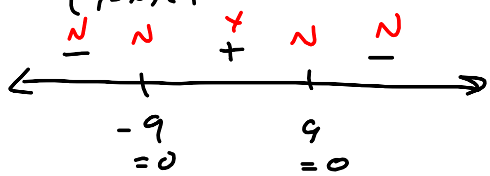
I think you're ready for this.

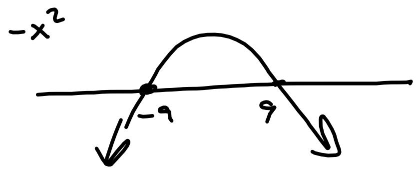
Find the domain of the function. (Enter your answer using interval notation.)

25 $f(x) = \dfrac{x}{\sqrt[4]{81 - x^2}}$

Need $81 - x^2 \geq 0$ AND $81 - x^2 \neq 0$ $\Rightarrow$ $81 - x^2 > 0$

$(9-x)(9+x) \geq 0$





$= \boxed{(-9, 9) = \mathcal{D}}$

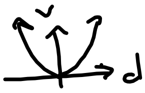
A verbal description of a function is given.

26 Let $V(d)$ be the volume of a sphere of diameter d. To find the volume, take the cube of the diameter, then multiply by π and divide by 6.

$V(d) = d^3 \cdot \pi \div 6 = \frac{\pi d^2}{6}$

(a) Find an algebraic representation for the function.

$v(d) = \frac{\pi d^2}{6}$



$V(\text{radius}) = \frac{4\pi r^3}{3}$

Note $d \geq 0$, so

$\mathcal{D} = [0, \infty)$

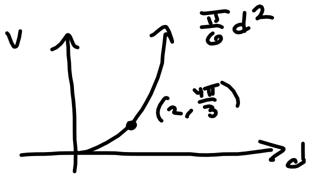
(b) Find a numerical representation for the function.

!

$\frac{\pi(2)^3}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$.

d	$V(d)$
2	$\frac{4\pi}{3}$
3	
4	
5	

(c) Find a graphical representation for the function.



27

Find the domain and range of f.

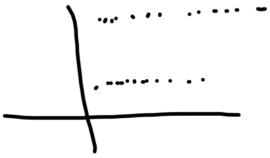
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 4 & \text{if } x \text{ is irrational} \end{cases}$$

Domain:

- ○ {1, 4}
- ○ all rational numbers
- ○ all irrational numbers
- ○ {1, all irrational numbers}
- ○ all real numbers

Rational #s = $\mathbb{Q} = \{ \frac{a}{b} \mid a,b \in \mathbb{Z},\ b \neq 0 \}$

$\mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$



Range:

- ○ {1, 4}
- ○ all rational numbers
- ○ all irrational numbers
- ○ {1, all irrational numbers}
- ○ all real numbers

28

The surface area S of a sphere is a function of its radius r given by

$$S(r) = 4\pi r^2.$$

Lexicon

$S = S(r)$ = surface area of sphere as a function of r

r = radius of sphere

(a) Find $S(4)$ and $S(5)$. (Round your answers to two decimal places.)

$S(4)$ = $4\pi(4^2) = 64\pi \approx$

$S(5)$ = $4\pi(5)^2 = 100\pi \approx$

(b) What do your answers in part (a) represent?

- ○ $S(4)$ represents the radius of a sphere with a surface area of 4, and $S(5)$ represents the radius of a sphere with a surface area of 5.
- ○ $S(4)$ represents the radius of a sphere with a surface area of 5, and $S(5)$ represents the radius of a sphere with a surface area of 4.
- ○ $S(4)$ represents the surface area of a sphere of radius 4, and $S(5)$ represents the surface area of a sphere of radius 5.
- ○ $S(5)$ represents the initial surface area, and $S(4)$ represents the final surface area.
- ○ $S(4)$ represents the surface area of a sphere of radius 5, and $S(5)$ represents the surface area of a sphere of radius 4.

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A hotel chain charges $115 each night for the first two nights and $92 for each additional night's stay. The total cost T is a function of the number of nights x that a guest stays.

(a) Complete the expressions in the following piecewise defined function.

$$T(x) = \begin{cases} 115x & \text{if } 0 \le x \le 2 \\ 230 + 92(x-2) & \text{if } x > 2 \end{cases}$$

$115x$ $(115)(2)=230$

$\begin{array}{r} 230 \\ 92 \\ \hline 322 \end{array}$ $\begin{array}{r} 92 \\ 3 \\ \hline 276 \end{array}$ $\begin{array}{r} 276 \\ 230 \\ \hline 506 \end{array}$

(b) Find $T(2)$, $T(3)$, and $T(5)$.

$T(2) = \$230$

$T(3) = \$322$

$T(5) = \$506$

Lexicon

Let $T = T(x)$ = total cost of hotel stay (in \$) as a function of x = # of nights of your stay.

(c) What do your answers in part (b) represent?

- ○ The cost of only the last night.
- ○ The number of nights one can stay for a certain cost.
- ○ The maximum number of nights a person can stay at the hotel.
- ○ The total cost of the stay.
- ○ The total cost of parking.

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