Section 1.8 Solving Inequalities

It's just like solving equations, almost. Things change when you divide by a negative number (The inequality flips.)

Also, when dealing with fractions involving variables in the denominator, your old "Clear fractions by multiplying by the LCD" won't cut it!

You need to go "old school," as I've said in my earlier videos, and NEVER throw away the LCD, because zeros in the denominator affect the sign of the expression.

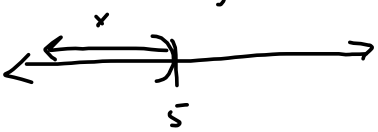
The basics

$3x - 7 < 8$

$+7 = +7$

$3x < 15$

$x < \frac{15}{3} = 5$



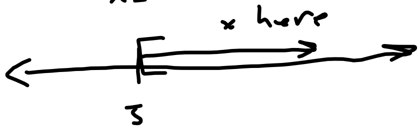
$= (-\infty, 5)$

$3x - 7 \geq 8$

$3x \geq 15$

$x \geq 5$

x here



$= [5, \infty)$

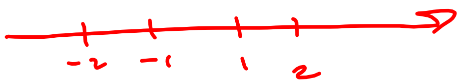
$-3x - 7 > 5$

$-3x > 12$ ← I need to see this

$x < \frac{12}{-3} = -4$

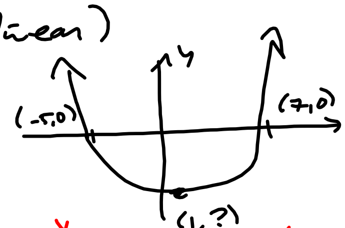
$1 < 2$

$-1 > -2$

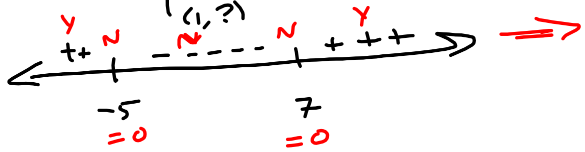


Managing Products & Quotients (Nonlinear)

$(x+5)(x-7) > 0$ → +



SIGN PATTERN



$\Rightarrow x \in (-\infty,-5) \cup (7,\infty)$

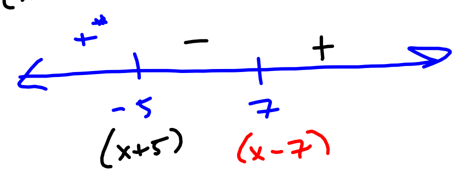
FACT: When a factor changes sign, so does the product.

$(x+5)(x-7)$

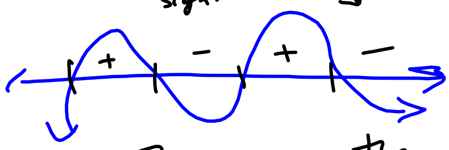
Intervals	Test		>0?
$(-\infty,-5)$	-6	$(-6+5)(-6-7) = (-1)(-13) = +13$	Yes
$(-5,7)$		ETC.	
$(7,\infty)$			

* by plugging in $x=-6$

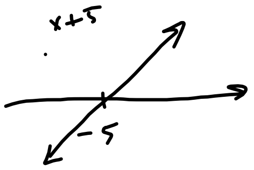
$(x+5)(x-7)$

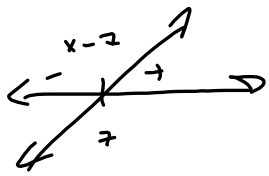


Use any trick to find ONE interval's sign.



Then manage the sign changes by looking at the factors in control

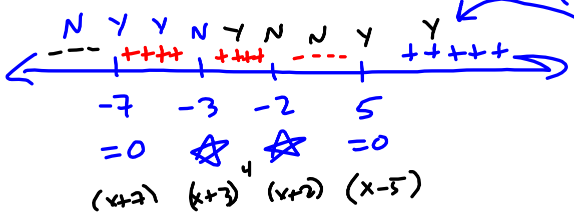




$$\frac{(x+7)(x-5)}{(x+2)(x+3)^4} \geq 0$$

$$\frac{(\text{Big}+7)(\text{Big}-5)}{(\text{Big}+2)(\text{Big}+3)^4} = \frac{(+)(+)}{(+)(+)^4} = +$$

≥ 0



$\Rightarrow x \in [-7,-3) \cup (-3,-2) \cup [5,\infty)$

$$\frac{(x+7)(x-5)}{(x+2)(x+3)^4}$$

Fill in each blank with an appropriate inequality sign.

(a) If $x < 6$, then $x - 3$ [--?--] 3.

1

(b) If $x \leq 6$, then $3x$ [--?--] 18.

(c) If $x \geq 3$, then $-3x$ [--?--] -9.

(d) If $x < -3$, then $-x$ [--?--] 3.

2 To solve the nonlinear inequality $\frac{x+3}{x-7} \leq 0$, we first observe that the numbers ____ (smaller value) and ____ (larger value) are zeros of the numerator and denominator. These numbers divide the real line into three intervals. Complete the table.

Interval	$(-\infty, \_\_)$	$(\_\_, \_\_)$	$(\_\_, \infty)$
Sign of $x + 3$	--?--	--?--	--?--
Sign of $x - 7$	--?--	--?--	--?--
Sign of $(x + 3)/(x - 7)$	--?--	--?--	--?--

Do any of the endpoints fail to satisfy the inequality?

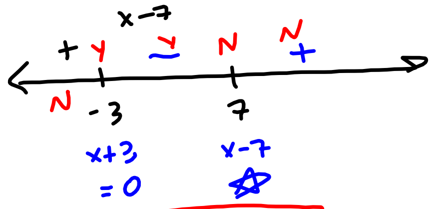
- ○ Yes
- ○ No

If so, which one(s)? (Enter your answers as a comma-separated list. If an answer does not exist, enter DNE.)

[ ]

Find the solution of the inequality. (Enter your answer using interval notation.)

$\frac{x+3}{x-7} \leq 0$



Just Like $(x+3)(x-7) \leq 0$

except $x = 7$ is BAD

≤ 0

Want "−" & "="

$\Rightarrow$ $x \in [-3, 7)$

What is a logical first step in solving the inequality?

(a) $5x \le 2$ ⇒ $x \le \frac{2}{5}$

3

- ◯ Multiply both sides of the inequality by 2.
- ◯ Divide both sides of the inequality by 2.
- ◯ Subtract 5 from both sides of the inequality.
- ◯ Divide both sides of the inequality by 5.
- ◯ Multiply both sides of the inequality by 5.

(b) $5x - 4 \ge 1$ ⇒ $5x \ge 5$

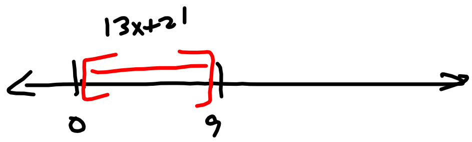
- ◯ Add 1 to both sides of the inequality.
- ◯ Add 4 to both sides of the inequality.
- ◯ Subtract 1 from both sides of the inequality.
- ◯ Subtract 5 from both sides of the inequality.
- ◯ Subtract 4 from both sides of the inequality.

(c) $|3x + 2| \le 9$

- ◯ Consider the two cases $|3x + 2| = 3x + 2$ and $|3x + 2| = -(3x + 2)$.
- ◯ Consider the two cases $|3x + 2| = 9$ and $|3x + 2| = -9$.
- ◯ Subtract 2 from both sides of the inequality.
- ◯ Rewrite the inequality as $3x + 2 \le 9$.
- ◯ Divide both sides of the inequality by 3.

See S1.9

$|3x+2| \le 9$ ⇒

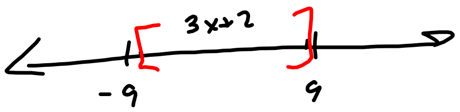


$3x+2 \le 9$ AND $3x+2 \ge -9$

See above:

$-(3x+2) \le 9$ ⇒

$3x+2 \ge -9$



Solve the linear inequality. Express the solution using interval notation.

4 $2x \le 9$

$(-\infty, \frac{9}{2}]$

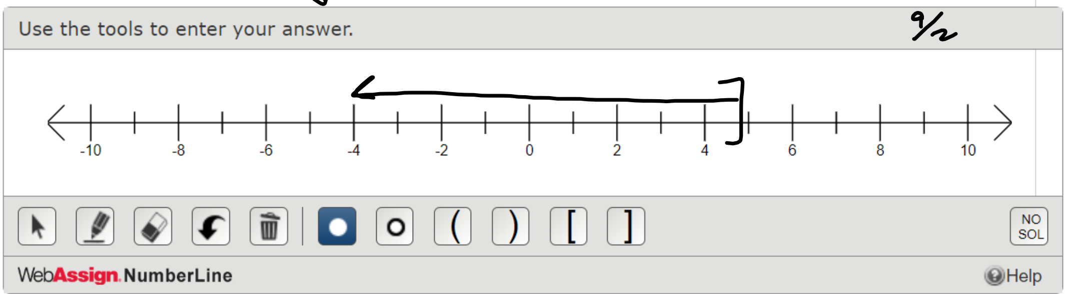
$2x \le 9$

$\frac{2x}{2} \le \frac{9}{2}$

$x \le \frac{9}{2} = 4.5$

$\frac{9}{2}$

Graph the solution set. Do this before that



Solve the linear inequality. Express the solution using interval notation.

5 $-4x \ge 18$

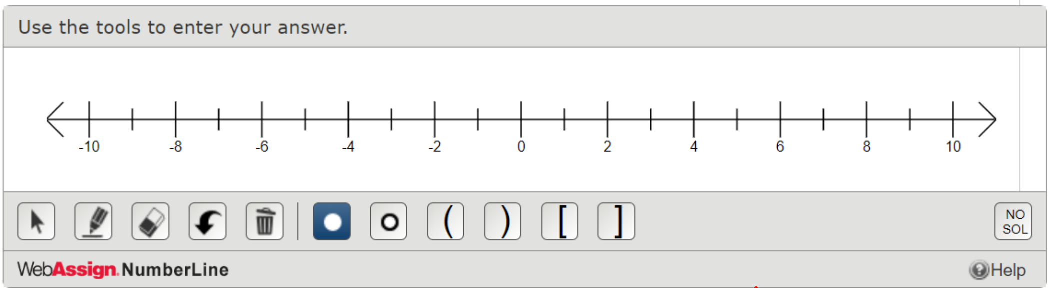
$-4x \ge 18$

$x \le \frac{18}{-4} = -\frac{9}{2}$

$-\frac{9}{2}$ $= (-\infty, -\frac{9}{2}]$

See Below for Poor Technique to avoid

Graph the solution set.



$\frac{-4x}{-4} \ge \frac{18}{-4} = -\frac{9}{2}$

$x \ge -\frac{9}{2}$

Didn't flip to "≤"!

Alternate: This is ok.

$-4x \ge 18$

$\frac{-4x}{-4} \ge \frac{18}{-4} = -\frac{9}{2}$

$x \ge -\frac{9}{2}$

$\frac{-4x}{-4} \le \frac{18}{-4}$

BAD B/C I can't see the original $-4x \ge 18$

$\frac{-4x}{-4} \ge \frac{18}{-4}$ ← NOT TRUE!

$x \le -\frac{9}{2}$ ← Does not follow from previous step

You told 2 lies to state the eventual truth. Doubly Bad!

Solve the linear inequality. Express the solution using interval notation.

6 $3x - 8 > 13$

Graph the solution set.

Solve the linear inequality. Express the solution using interval notation.

7 $6 - x \geq 3$

Graph the solution set.

Solve the linear inequality. Express the solution using interval notation.

8 $1 - 3x \leq -17$

Graph the solution set.

Solve the linear inequality. Express the solution using interval notation.

9 $\left(\frac{2}{5}x + 9 < \frac{6}{5} - 2x\right) \longrightarrow$

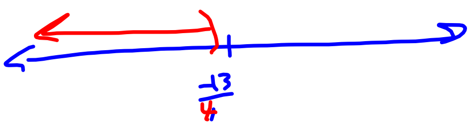
$2x + 45 < 6 - 10x$

$10x - 45 \quad -45 \quad 10x$

$12x < -39$

$x < -\frac{39}{12} = -\frac{13}{4}$

Graph the solution set.



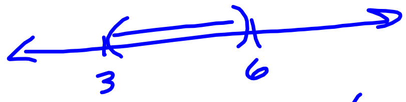
$= \boxed{\left(-\infty, -\frac{13}{4}\right)}$

Solve the linear inequality. Express the solution using interval notation.

10 $-1 < 2x - 7 < 5$

$-1 < 2x - 7 < 5$
$+7 = \quad +7 \quad = +7$
$6 < 2x < 12$
$3 < x < 6$

Graph the solution set.



$3 < x$ AND $x < 6$
→ Restriction.

Solve the linear inequality. Express the solution using interval notation.

11 $1 < 3x + 7 \leq 22$

Graph the solution set.

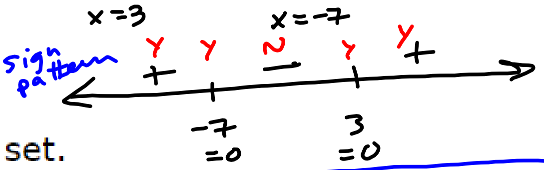
Solve the nonlinear inequality. Express the solution using interval notation.

12 $(x-3)(x+7) \geq 0$

x^2 + smaller $\xrightarrow{x \to \text{Big}}$ BIG, Positive

$x-3=0$ $x+7=0$

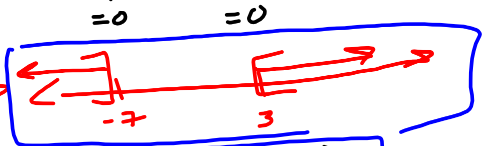
$x=3$ $x=-7$



want ≥ 0
"+" or "=0"

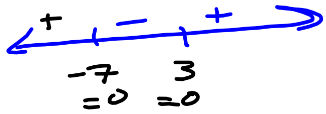
Graph the solution set.

LINE GRAPH ⇒



$= (-\infty, -7] \cup [3, \infty)$

$(\text{Big}-3)(\text{Big}+7) \approx (\text{Big})(\text{Big}) = (+)(+) = +$



Analyze the sign

Interval	Test Value	
$(-\infty,-7)$	-8	$(-8-3)(-8+7) = (-)(-) = +$
$(-7,3)$	2	$(2-3)(2+7) = (-)(+) = -$
$(3,\infty)$	4	$(4-3)(4+7) = (+)(+) = +$

Solve the nonlinear inequality. Express the solution using interval notation.

13 $x(2x + 3) \geq 0$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

14 $x(6 - 7x) \leq 0$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

15 $x^2 - 4x - 32 \leq 0$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

16 $2x^2 + x \geq 6$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

17 $5x^2 + 7x \geq 3x^2 + 30$

Graph the solution set.

18

Solve the nonlinear inequality. Express the solution using interval notation.

$$x^2 > 4(x + 8)$$

Graph the solution set.

19

Solve the nonlinear inequality. Express the solution using interval notation.

$$x^2 + 2x > 35$$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

20 $(x + 1)(x - 4)(x - 9) \leq 0$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

21 $(x - 6)(x - 3)(x + 2) > 0$

Graph the solution set.

Solve the nonlinear inequality. Express the solution using interval notation.

22 $(x - 8)(x + 7)^2 < 0$

Graph the solution set.

23

Solve the nonlinear inequality. Express the solution using interval notation.

$$x^3 - 25x > 0$$

Graph the solution set.

24

Solve the nonlinear inequality. Express the solution using interval notation.

$$\frac{x - 5}{x + 4} \geq 0$$

Graph the solution set.

25

Solve the nonlinear inequality. Express the solution using interval notation.

$$\frac{3x + 15}{x - 7} < 0$$

Graph the solution set.

26 Solve the nonlinear inequality. Express the solution using interval notation.

$$\frac{5+x}{5-x} \ge 1$$

Do NOT CLEAR FRACTIONS!!!!
WRITE EVERYTHING OVER THE LCD AND GET ZERO ON THE RIGHT HAND SIDE!!!

LCD = 5 − x

$\frac{5+x}{5-x} \ge \left(\frac{1}{1}\right)\left(\frac{5-x}{5-x}\right) = \frac{5-x}{5-x}$

$-\frac{5-x}{5-x} \qquad = -\frac{5-x}{5-x}$

Graph the solution set.

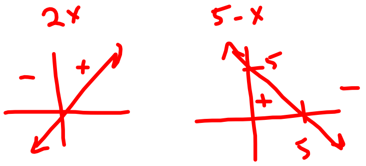
$\frac{5+x}{5-x} - \frac{5-x}{5-x} \ge 0$

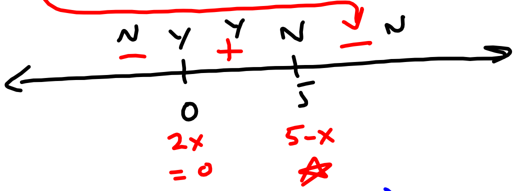
$\frac{5+x-(5-x)}{LCD} \ge 0$

$\frac{5+x-5+x}{LCD} \ge 0$

$\frac{2x}{5-x} \ge 0$ $\quad 2x = 0 \Rightarrow x = 0$ $\quad 5 - x = 0 \Rightarrow x = 5$

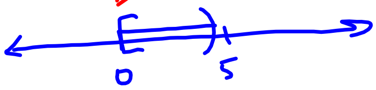
x = BIG ⇒ $\frac{2BIG}{5-BIG} = \frac{+}{-} = -$





WANT ≥ 0
"+" or "= 0"

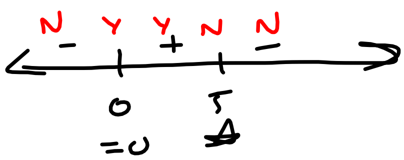
You may also use test values in each subinterval.



⇒ x ∈ … = [0, 5)

	Test	
(−∞, 0)	x = −1	$\frac{2(-1)}{5-(-1)} = \frac{-}{+} = -$
(0, 5)	x = 1	$\frac{2(1)}{5-1} = \frac{2}{4} = +$
(5, ∞)	x = 6	$\frac{2(6)}{5-6} = \frac{12}{-1} = -$

SIGN PATTERN



≥ 0 "+"

etc.

27 Solve the nonlinear inequality. Express the solution using interval notation.

$$\frac{x}{x+1} > 4x$$

Same skill as #26.

Graph the solution set.

28 Solve the nonlinear inequality. Express the solution using interval notation.

$$1 + \frac{20}{x+1} \le \frac{20}{x}$$

LCD: $x(x+1)$

Domain: $x \in \mathbb{R} \setminus \{0, -1\}$

$(x \ne 0$ & $x+1 \ne 0)$

$$\frac{1}{1} \cdot \frac{(x(x+1))}{x(x+1)} + \frac{20}{x+1} \cdot \frac{x}{x} - \frac{20}{x} \cdot \frac{(x+1)}{(x+1)} \le 0 \Rightarrow$$

Graph the solution set.

$$\frac{x^2+x+20x-20x-20}{LCD} = \frac{x^2+x-20}{LCD}$$

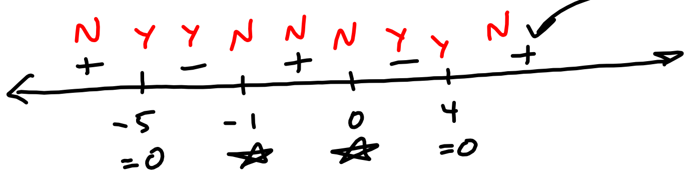
$$= \frac{(x+5)(x-4)}{x(x+1)} \le 0$$

$$\frac{(BIG+5)(B-4)}{BIG(BIG+1)} \approx \frac{(BIG)(BIG)}{(BIG)(BIG)} = \frac{(+)(+)}{(+)(+)} = +$$

Important points

Num. $(x+5)(x-4) = 0 \Rightarrow x = -5, 4$ "= 0"

Denom. $x(x+1) = 0 \Rightarrow x = -1, 0$ "✗"



WANT ≤ 0, i.e., "−" OR "= 0"

$= [-5, -1) \cup (0, 4]$ → Interval sol'n.

$x \in [-5, -1) \cup (0, 4] = \{x \mid -5 \le x < -1 \text{ OR } 0 < x \le 4\}$

29 Solve the nonlinear inequality. Express the solution using interval notation.

$$\frac{x+3}{x+4} < \frac{x-1}{x-3}$$

$\Rightarrow$ $\mathcal{D} = \mathbb{R} \setminus \{-4, 3\}$

LCD $= (x+4)(x-3)$

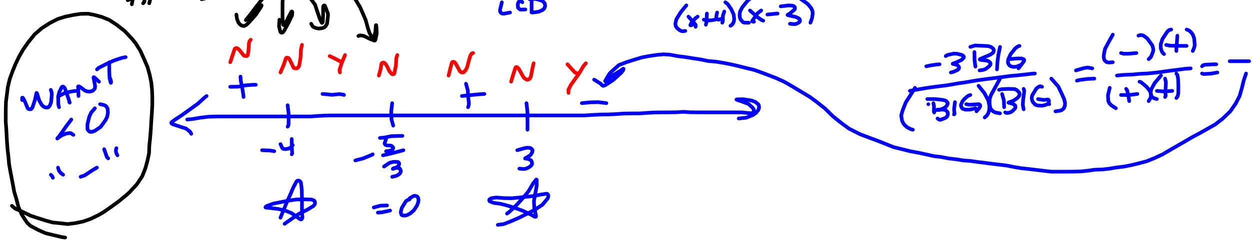
$\Rightarrow \left(\frac{x+3}{x+4}\right)\left(\frac{x-3}{x-3}\right) - \left(\frac{x-1}{x-3}\right)\left(\frac{x+4}{x+4}\right) < 0 \Rightarrow$

Graph the solution set.

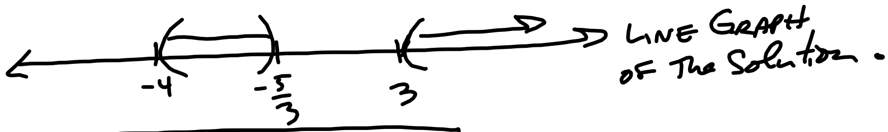
$\frac{x^2-9-(x^2+3x-4)}{\text{LCD}} = \frac{x^2-9-x^2-3x+4}{\text{LCD}}$

$= \frac{-3x-5}{\text{LCD}} = \frac{-3(x+\frac{5}{3})}{(x+4)(x-3)} < 0$

Analyze:



$\Rightarrow$



$= \boxed{(-4, -\frac{5}{3}) \cup (3, \infty)}$

30

Recognize the type of inequality and solve the inequality by an appropriate method. Express the answer using interval notation.

$$\frac{x}{x+3} > 5$$

Graph the solution set.