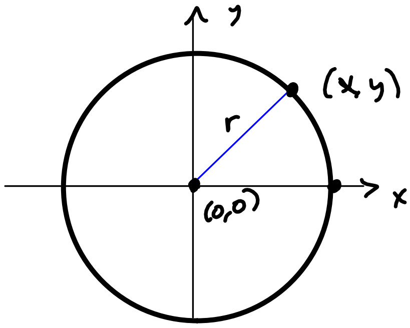
Section 1.3 - Circles

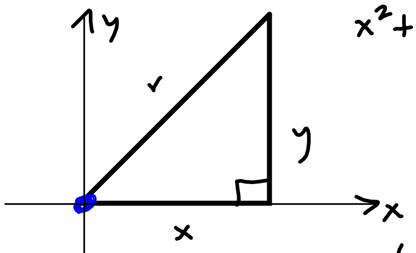
We rely very heavily on the Distance Formula from Section 1.1.

The Standard Form of the equation of a Circle of radius $r$ in the $xy$-plane, with its center at the origin (0, 0) is given by

$$x^2+y^2=r^2$$

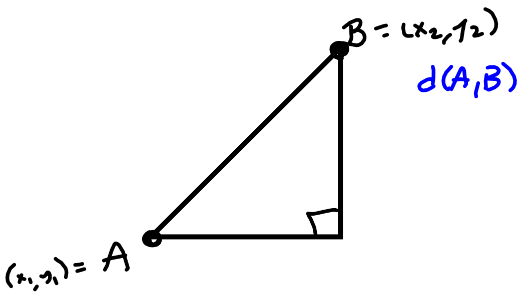
and any point $(x,y)$ on its graph satisfies the equation.



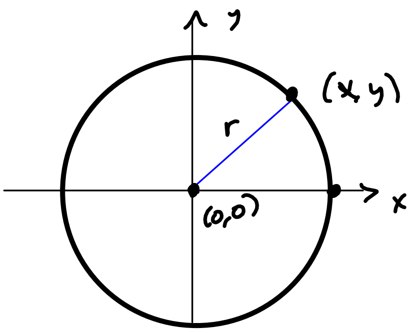

$x^2+y^2=r^2$

Everything's $r$ units from $(0,0)$




$d(A,B)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

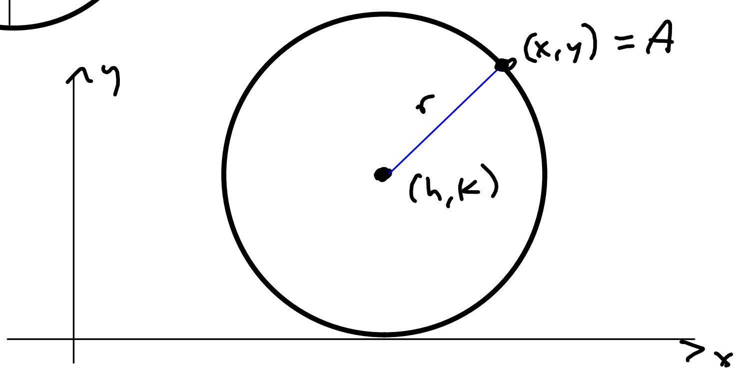
The distance from $A = (x, y)$ to the origin is $r$ :




$$\sqrt{(x-0)^2+(y-0)^2}=r$$

$$\left(\sqrt{x^2+y^2}\right)^2=(r)^2$$

$x^2+y^2=r^2$ & done!



In general, the equation of a circle of radius *r* and center (*h*, *k*) is given by:

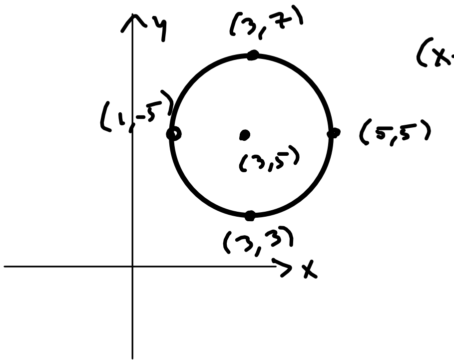
(x,y) is on the circle of radius r w/ center (h,k)

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

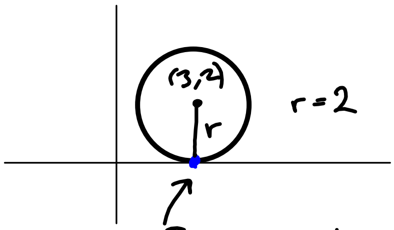
$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

center (3,5), radius 2

$$(x-3)^2+(y-5)^2=2^2$$



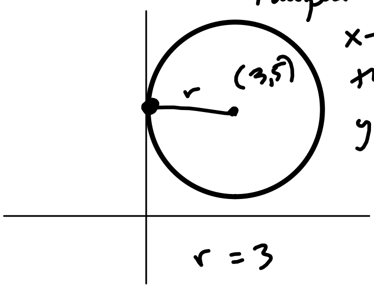

Tangent to the x-axis




r=2

Just touches & so the y-coordinate of the center gives you the length of the radius

Tangent to y-axis.
x-coord of the center gives you the radius




r = 3

Recall: $(u+w)^2 = u^2 + 2uw + w^2$

Proof: $(u+w)^2 = (u+w)(u+w) = u^2 + uw + wu + w^2$

$= u^2 + uw + uw + w^2 = u^2 + 2uw + w^2$

Learn to complete the square in as short a time as possible.

ANY expression

$$x^2 + bx$$

may be written in the form

$$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Proof:

$u = x$

$w = \frac{b}{2}$

$\left(x + \frac{b}{2}\right)^2 = (u+w)^2 = u^2 + 2uw + w^2$

$= x^2 + 2(x)\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2$

$= x^2 + bx + \left(\frac{b}{2}\right)^2$ & so

$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = x^2 + bx$

We'll need this for #s 11, 12, ... in §1.3

WRITE

Show that the equation represents a circle by rewriting it in standard form.

#12 $x^2 + y^2 + 8x - 4y + 19 = 0$

$x^2 + 8x \quad + y^2 - 4y \quad = -19$

$\frac{8}{2} = 4 \rightsquigarrow 4^2 = 16 \qquad \frac{4}{2} = 2 \rightsquigarrow 2^2 = 4$

$(x+4)^2 - (4)^2 + (y-2)^2 - (2)^2 = -19$

$(x+4)^2 + (y-2)^2 = -19 + 16 + 4$

$\boxed{(x+4)^2 + (y-2)^2 = 1}$

$(h,k) = (-4,2)$, $r = \sqrt{1} = 1$

Find the center and radius of the circle.

center $(x, y) = (-4, 2)$

radius $r = 1$

In Practice: Add $\left(\frac{b}{2}\right)^2$ to Both sides.

$x^2 + y^2 + 8x - 4y + 19 = 0$

$x^2 + 8x + 4^2 \; y^2 - 4y + 2^2 = -19 + 16 + 4$

$\frac{8}{2} = 4 \rightsquigarrow 4^2 = 16 \qquad \frac{4}{2} = 2 \rightsquigarrow 2^2 = 4$

$(x+4)^2 + (y-2)^2 = 1$

In the sequel:

Writing Project #2

$$f(x) = x^2 + 8x$$
$$= x^2 + 8x + 4^2 - 16$$
$$= (x+4)^2 - 16$$

Now we see $f(x)$ as $x^2$ shifted left 4 and down 16.

Find the center and radius of the circle.

1 $x^2 + y^2 = 81 = r^2$

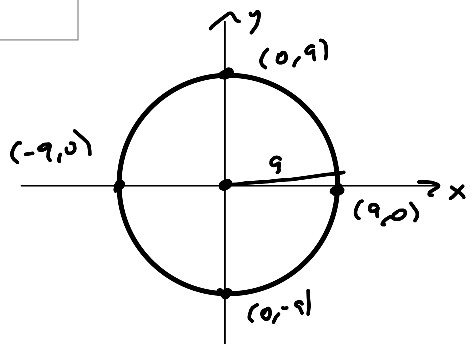
$(h,k) = (0,0)$

$r = \sqrt{81} = 9$

center $(x, y) = ($ 0,0 $)$

radius $r =$ 9

Sketch its graph.




Find the center and radius of the circle.

$(x - 4)^2 + y^2 = 4$

$(x-4)^2 + (y-0)^2 = 4 = 2^2 = r^2$

$x - h = x - 4$
$-h = -4$
$h = 4$

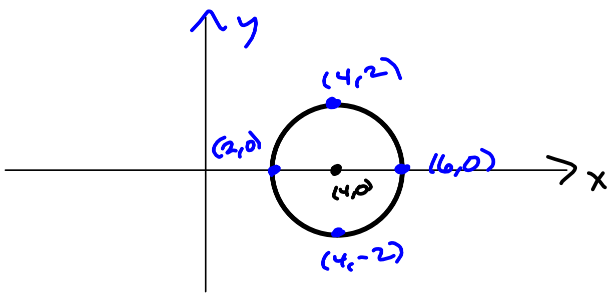
$y - k = y - 0$
$-k = 0$
$k = 0$

center $(x, y) = ($ 4,0 $)$

2

radius 2

Sketch the graph of the circle.



Find the center and radius of the circle.

3 $(x+2)^2 + (y-1)^2 = 16$

center $(x, y) = \left( \boxed{\phantom{00}} \right)$

radius $r = \boxed{\phantom{00}}$

Sketch its graph.

Find the center and radius of the circle.

4 $(x+4)^2 + (y+2)^2 = 9$

center $(x, y) = \left( \boxed{\phantom{00}} \right)$

radius $r = \boxed{\phantom{00}}$

Sketch its graph.

Find an equation of the circle that satisfies the given conditions. (Use the variables $x$ and $y$.)

5 Center $(-1, 3)$, radius 1

Find an equation of the circle that satisfies the given conditions. (Use the variables $x$ and $y$.)

6 Center $(-8, -4)$, radius 2

Find an equation of the circle that satisfies the given conditions.

7 Center at the origin; passes through (1, 6)

$r = \sqrt{(1-0)^2 + (6-0)^2} = \sqrt{6^2+1^2} = \sqrt{36+1} = \sqrt{37}$

$(x-1)^2 + (y-6)^2 = (\sqrt{37})^2$

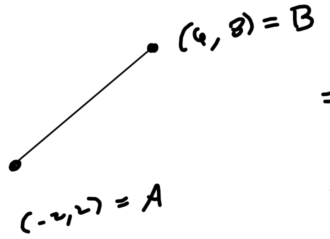
Could've just done

$r^2 = 1^2+6^2 = 37$

$(x-1)^2 + (y-6)^2 = 37$

Find an equation of the circle that satisfies the given conditions.

8 Endpoints of a diameter are $P(-2, 2)$ and $Q(6, 8)$




$d(A,B)$

$= \sqrt{(6-(-2))^2 + (8-2)^2} = \sqrt{8^2+6^2} = \sqrt{64+36}$

$= \sqrt{100} = 10 =$ diameter $\Rightarrow$

radius $= \boxed{r = 5}$

Midpoint $(A,B) =$ Center $= \left(\frac{-2+6}{2}, \frac{2+8}{2}\right) = \left(\frac{4}{2}, \frac{10}{2}\right)$

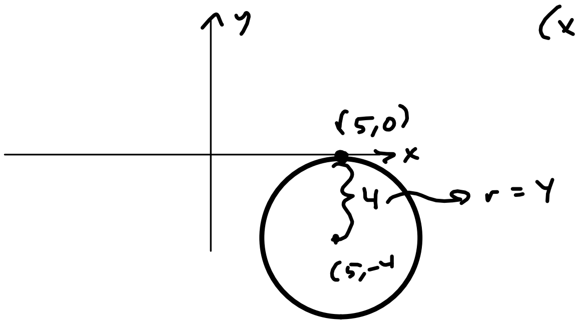
$= \boxed{(2,5) = (h,k)}$

$(x-h)^2 + (y-k)^2 = r^2$

$\Rightarrow (x-2)^2 + (y-5)^2 = 5^2$

Find an equation of the circle that satisfies the given conditions.

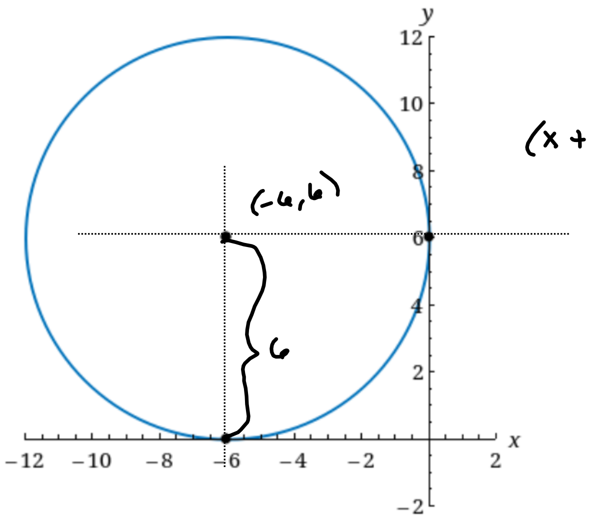
**9** Center (5, −4); tangent to the $x$-axis




$(x-5)^2+(y+4)^2=16$

Find the equation of the circle shown in the figure.

**10**




$(x+6)^2+(y-6)^2=36$

Show that the equation represents a circle by rewriting it in standard form.

**11** $x^2 + y^2 - \frac{1}{6}x + \frac{1}{6}y = \frac{1}{72}$

You Probably Should See #12.

This one's mechanics are more complicated. I got #11 and #12 out of sequence because WebAssign presented them in this order, and I didn't catch it in time..

Find the center and radius of the circle.

center $(x, y) = (\quad)$

radius $r =$

Click Here for Completing the Square for Circles video.

$$x^2 - \frac{1}{6}x + \left(\frac{1}{12}\right)^2 + y^2 + \frac{1}{6}y + \left(\frac{1}{12}\right)^2 = \frac{1}{72} + \frac{1}{144} + \frac{1}{144}$$

$$\frac{\frac{1}{6}}{2} = \frac{\frac{1}{6}}{\frac{2}{1}} = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12} \rightsquigarrow \left(\frac{1}{12}\right)^2 = \frac{1}{144}$$

$$\frac{1}{72} + \frac{2}{144} =$$

$$\frac{1}{72} + \frac{1}{72} = \frac{2}{72} = \frac{1}{36}$$

$$= \left(\frac{1}{6}\right)^2$$

$$\left(\frac{1}{6}\right) \div 2 = \frac{1}{12} \rightsquigarrow \left(\frac{1}{12}\right)^2 = \frac{1}{144}$$

$$\left(x - \frac{1}{12}\right)^2 + \left(y + \frac{1}{12}\right)^2 = \frac{1}{36} = \left(\frac{1}{6}\right)^2$$

Show that the equation represents a circle by rewriting it in standard form.

**12** $x^2 + y^2 + 8x - 4y + 19 = 0$

$x^2 + 8x \quad + y^2 - 4y \quad = -19$

$(x+4)^2 + (y-2)^2 = -19 + 20$

$(x+4)^2 + (y-2)^2 = 1$

See the intro here:

Find the center and radius of the circle.

center $(x, y) = (-4, 2)$

radius $r = 1$

Click Here for Intro Video, where #12 is an example!

Show that the equation represents a circle by rewriting it in standard form.

**13** $6x^2 + 6y^2 - 5x = 0$

Divide by 6:

$x^2 - \frac{5}{6}x \quad + y^2 = 0$

$x^2 - \frac{5}{6}x + \left(\frac{5}{12}\right)^2 + y^2 = \frac{25}{144}$

$\left(x - \frac{5}{12}\right)^2 + y^2 = \frac{25}{144} = \left(\frac{5}{12}\right)^2$

Find the center and radius of the circle.

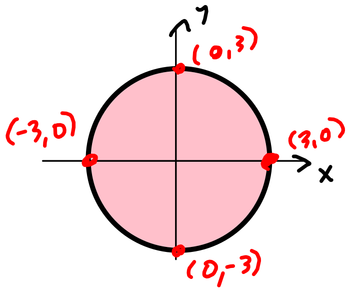
center $(x, y) = \left(\frac{5}{12}, 0\right)$

radius $r = \frac{5}{12}$

Sketch the region given by the set.

**14** $\{(x, y) \mid x^2 + y^2 \le 9\}$

We're inside or on the circle of radius $r = 3$, centered at the origin.




Show that the equation represents a circle by rewriting it in standard form.

**15** $5x^2 + 5y^2 + 10x - y = 0$

See #13

Find the center and radius of the circle.

center $(x, y) = (\quad)$

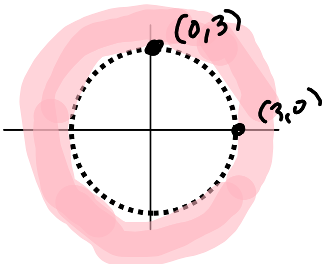
radius $r =$

Sketch the region given by the set.

**16** $\{(x, y) \mid x^2 + y^2 > 9\}$

This is just like #14, only it's ">" instead of "≤".

So we're OUTSIDE the circle, but NOT on the boundary, so we need a dotted line on the boundary of our shaded region.



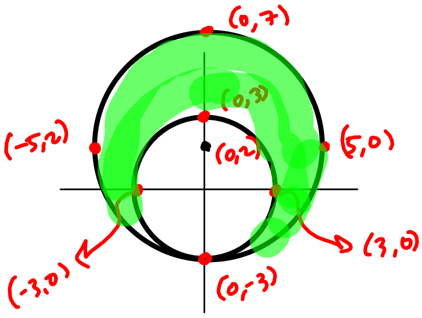

**17** Find the area of the region that lies outside the circle $x^2 + y^2 = 9$ ($C_1$) but inside the circle $x^2 + y^2 - 4y - 21 = 0$ ($C_2$).

$(0,0)$, $r = 3$

$x^2 + y^2 - 4y + 2^2 = 21 + 4$

$x^2 + (y-2)^2 = 25$

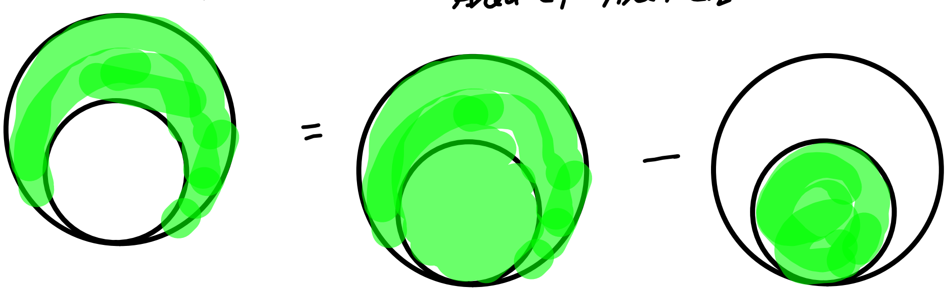
$(0,2)$, $r = 5$




$C_2 \subseteq C_1$

Area outside $C_2$ & inside $C_1$ =

Area $C_1$ − Area $C_2$



$r = 5$ $\qquad$ $r = 3$

$\pi(5)^2 - \pi(3)^2$

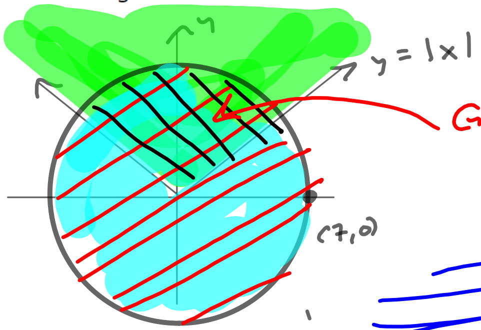
$25\pi - 9\pi = 16\pi$

Sketch the region in the coordinate plane that satisfies both the inequalities $x^2 + y^2 \le 49$ and $y \ge |x|$.

**18**

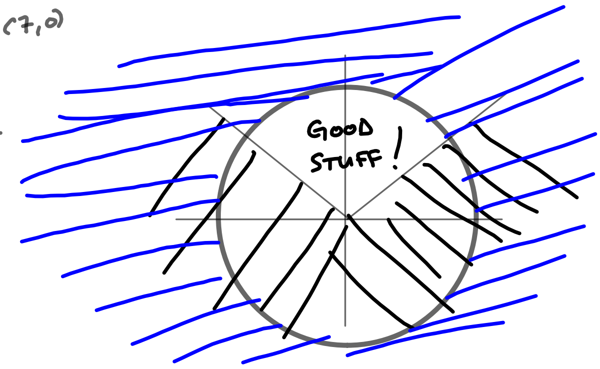
What is the area of this region?

$y \ge |x|$ :




GOOD STUFF

$x^2+y^2 \le 49$




My Way

Scratch out Bad Stuff

It's $\frac{1}{4}$ the area of the circle

$\frac{\pi(7)^2}{4} = \frac{49\pi}{4}$!

**19**

(a) Find the radius $r$ of each circle in the pair and the distance between their centers; then use this information to determine whether the circles intersect.

See if the sum of the radii is greater than the distance between the centers!

(i) $(x - 3)^2 + (y - 1)^2 = 9$ $A = (3,1), r_1 = 3$

$r =$ 3

$(x - 7)^2 + (y - 4)^2 = 25$ $B = (7,4), r_2 = 5$

$r =$ 5

$5+3 = \boxed{8 = r_1 + r_2}$

Find the distance between their centers.

5

$d(A,B) = \sqrt{(3-7)^2 + (1-4)^2} = \sqrt{4^2+3^2}$

$= \sqrt{16+9} = \sqrt{25} = 5$

Yes! $r_1 + r_2 = 8 > 5$!

Do the circles intersect?

- Yes
- No

(ii) $x^2 + (y - 5)^2 = 16$ $(h,k) = C = (0,5)$

$r =$ 4

$(x - 5)^2 + (y - 17)^2 = 49$ $(h,k) = D = (5,17)$

$r =$ 7

$d(C,D) = \sqrt{(0-5)^2 + (5-17)^2} = \sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$4+7 = 11 < 13 \Rightarrow$ NO

Find the distance between their centers.

13

Do the circles intersect?

- Yes
- No

(iii) $(x - 4)^2 + (y + 1)^2 = 1$ $(4,-1) = E$

$r =$ 1

$(x - 2)^2 + (y - 2)^2 = 49$ $F = (2,2)$

$r =$ 7

$d(E,F) = \sqrt{(4-2)^2 + (-1-2)^2}$

$= \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

$3 < \sqrt{13} < 4$

$9 < 13 < 16$

$r_1 + r_2 = 1+7 = 8 > 4 > \sqrt{13}$

so, yes

Find the distance between their centers.

$\sqrt{13}$

Do the circles intersect?

- Yes
- No

(b) How can you tell, just by knowing the radii of two circles and the distance between their centers, whether the circles intersect? Write a short paragraph describing how you would decide this, and draw graphs to illustrate your answer.

Move a distance from the center of the first equal to the radius of the 1st along line segment ($\overline{AB}$, $\overline{CD}$, or $\overline{EF}$, above)

Do the same for 2nd circle