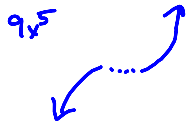
$f(x) = 9x^5 - 75x^4 - 224x^3 + 2172x^2 + 1511x + 255$

1. (2 pts) Describe the end behavior of f with a simple graphic.



(2 pts) Use Descartes' Rule of Signs to determine the *possible* number of positive and negative zeros.

$$f(x) = \underbrace{9x^5 - 75x^4}_{1} \underbrace{- 224x^3 + 2172x^2}_{2} + 1511x + 255$$

2 or 0 positive roots (zeros)

$$f(-x) = -9x^5 \underbrace{- 75x^4 + 224x^3}_{1} \underbrace{+ 2172x^2 - 1511x}_{2} \underbrace{- 1511x + 255}_{3}$$

3 or 1 negative roots.

So, at least one negative root, so look for those, first!

$$f(x) = 9x^5 - 75x^4 - 224x^3 + 2172x^2 + 1511x + 255$$

q's p's

3. (2 pts) Use the Rational Zeros Theorem to determine the *possible* rational zeros (roots) of f.

$\frac{p\text{'s}}{q\text{'s}}$ will be $\frac{\text{factors of } 255}{\text{" " } 9}$

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,

3 | 255
5 | 85
17
p's

3 | 9
3
q's

$\pm 1, \pm 3, \pm 5, \pm 17, \pm 15, \pm 51, \pm 85, \pm 255$

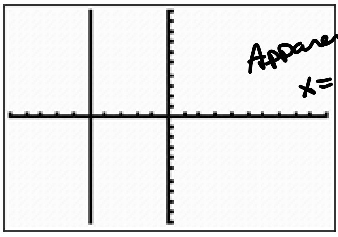
$\pm\frac{1}{3}, \cancel{\pm\frac{3}{3}}, \pm\frac{5}{3}, \pm\frac{17}{3}, \cancel{\pm\frac{15}{3}}, \cancel{\pm\frac{51}{3}}, \pm\frac{85}{3}, \cancel{\pm\frac{255}{3}}$

$\pm\frac{1}{9}, \cancel{\pm\frac{3}{9}}, \pm\frac{5}{9}, \pm\frac{17}{9}, \cancel{\pm\frac{15}{9}}, \cancel{\pm\frac{51}{9}}, \pm\frac{85}{9}, \cancel{\pm\frac{255}{9}} = \cancel{\frac{85}{3}}$

Your teacher's nice. Check out the integers in top row, 1st!

Also, since it's take-home, use a grapher!

4. (2 pts) Using the information, above, find all real zeros of f. Finding all zeros includes finding the multiplicity of each. This means performing multiple synthetic divisions. Always check for multiplicity greater than 1 with another synthetic division, just in case.



Apparently $x=-5$ works.

$9x^5 - 75x^4 - 224x^3 + 2172x^2 + 1511x + 255 = f(x)$

-5	9	-75	-224	2172	1511	255
		-45	600	-1880	-1460	-255
-5	9	-120	376	292	51 *	0 sweet!
		-45	825	$-$BIG	$-$HUGE	
	9	-165	1201	$-$BIG	No	

* $9x^4 - 120x^3 + 376x^2 + 292x + 51$ is Depressed Polynomial

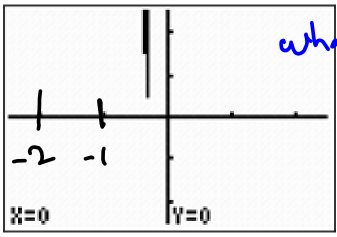
Never go back to original

This is where we are

$f(x) = (x+5)(9x^4 - 120x^3 + 376x^2 + 292x + 51)$

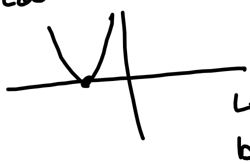
Now, my graph isn't showing everything in [-10, 10] window.

But I know the teacher. I'm gonna zoom in closer to the y-axis?



what's this saying, when I'm in a $[-25,25] \times [-25,25]$ window?

Looks like this is happening & calculator can't handle it,



Let's look at the list between -1 & 0

Try $x = -\frac{1}{3}$: $f(x) = (x+5)(9x^4 - 120x^3 + 376x^2 + 292x + 51)$

$-\frac{1}{3}$	9	-120	376	292	51
		-3	41	-139	-51
$-\frac{1}{3}$	9	-123	417	153	0 sweet!
		-3	42	-153	
$-\frac{1}{3}$	9	-126	459	0 sweet!	
		-3	Nope		
	9	-129			

Part of final answer.

here's what we have: $\boxed{\left(x+\frac{1}{3}\right)^2 (x+5)(9x^2 - 126x + 459) = f(x)}$

$9x^2 - 126x + 459 = 0$

$\Rightarrow 3x^2 - 42x + 153 = 0$

$\Rightarrow x^2 - 14x + 51 = 0$ to get rid of common factor of 9.

$x^2 - 14x + 7^2 - 49 + 51 = (x-7)^2 + 2 = 0 \Rightarrow$

$(x-7)^2 = -2$

$(x-7) = \pm\sqrt{-2} = \pm i\sqrt{2}$

$\boxed{x = 7 \pm i\sqrt{2}}$ non-real.

I shoulda/coulda done $b^2 - 4ac = 14^2 - 4(1)(51)$
$= 196 - 204 < 0$ to know the real stuff is done & written-up #4, but I did #5 in the process of making sure #4 was done, $9x^2 - 126x + 459$ is irreducible over the reals.

$\boxed{\text{so \#4 ; } x = -\tfrac{1}{3}, \underbrace{m = 2}_{\text{multiplicity of...}} ; 5, m = 1}$

5. (2 pts) From your work, above, factor f over the real numbers. This will involve an irreducible quadratic factor that your grapher has no way of helping you to see. without the synthetic divisions in #4, bringing you closer and closer, step by step, to the irreducible quadratic.

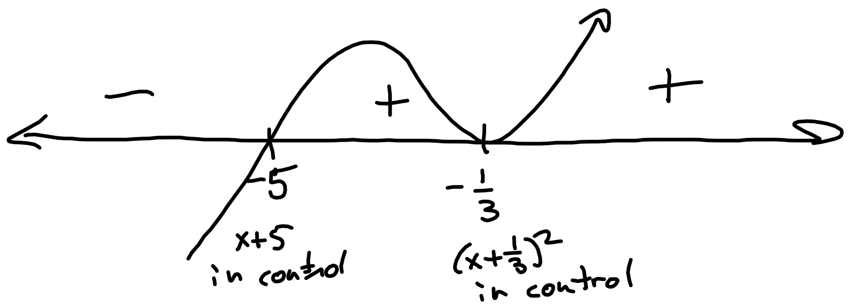
See answer to #4. I factored it, true, explaining stuff to Brice.

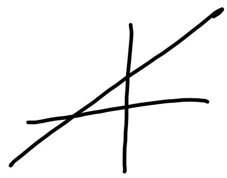
$(x+\frac{1}{3})^2(x+5)(9x^2-126x+459)=f(x)$

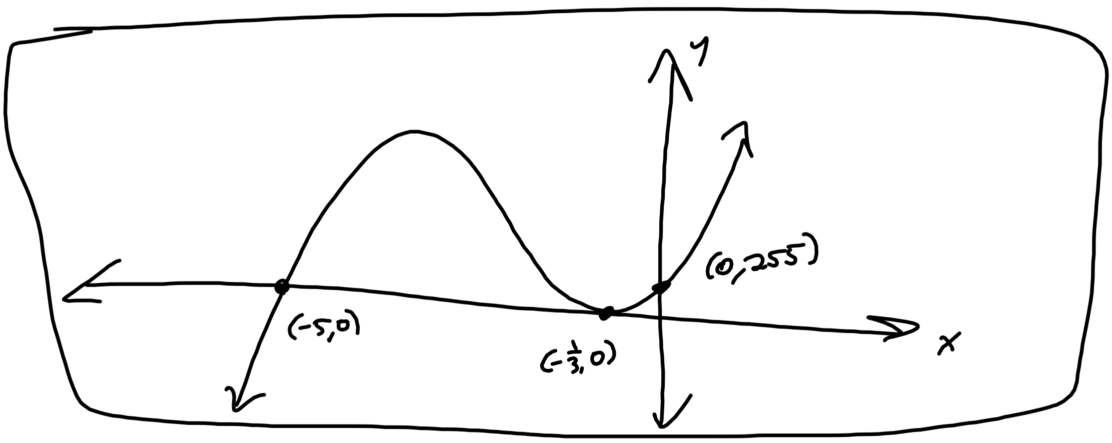
I got this all done before I knew this was #5 answer. I was just trying to show how we were breaking down the polynomial by splitting off factors.

6. (2 pts) Give a rough sketch of f from all of the above information. This is an *art* whose essence is really only found in my videos. If you're too tied to your grapher's output, you'll not capture the real essence of what's going on, or the key features I'm always looking for. Your picture will be more "vertical" than it should be.

Sign Patterns.







7. (2 pts) Now we've covered everything *real* about f. Let's use that work to find *all* the roots of f and *split f into linear factors*. 5 roots are *guaranteed by the Fundamental Theorem of Algebra,* and we have found the 3 real ones. The other 2 are nonreal, hiding inside the irreducible quadratic polynomial that remains as the last, very very depressed piece that's not broken all the way down in #5. Now do your quadratic equation thing to *find* the 2 nonreal roots. *Finally*, apply the Factor Theorem to *all* the above work, and represent f as a product of linear factors, $f(x)=a(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_w)^{m_w}$. Don't forget the leading coefficient, a.

This wrings (almost) every useful drop of the Theorems on Polynomials out of f, so now on to Rational Functions, which are *quotients* of polynomials!

$\boxed{x=7\pm i\sqrt{2}}$ are non-real roots &

$$\boxed{f(x)=9\left(x+\tfrac{1}{3}\right)^2(x+5)\left(x-(7+i\sqrt{2})\right)\left(x-(7-i\sqrt{2})\right)}$$

It's just natural to finish all #7 @ the end of #4

8. (5 pts) Sketch the graph of $R(x)=\dfrac{6x^2+11x-35}{x^2+x-20}$, showing all intercepts, asymptotes, and capturing the *essential features* of the shape of the graph. If you're a slave to your grapher, and oblivious to the features I'm looking for, it'll jump off the page at me (and be bad).

Domain & x-int., i.e., factor top & bottom.

$-(3)(2)(5)(7)$

$6x^2+21x-10x-35$

$= 3x(2x+7)-5(2x+7) = (2x+7)(3x-5)$

$R(x) = \dfrac{(2x+7)(3x-5)}{(x+5)(x-4)}$

$\mathcal{D} = \mathbb{R}\setminus\{-5,4\}$ No $x+5$ or $x-4$ upstairs so no holes, just vertical asymptotes : $\boxed{x=-5,\ x=4}$ V.A.

x-int: $R(x)=0 \Rightarrow (2x+7)(3x-5)=0 \Rightarrow$

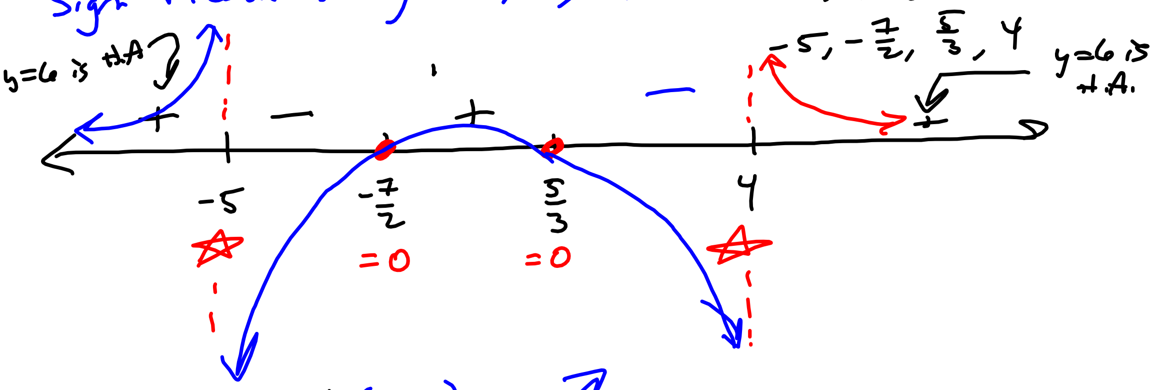
$x\in\{-\frac{7}{2},\frac{5}{3}\} \Rightarrow \boxed{(-\frac{7}{2},0),\ (\frac{5}{3},0)\ \text{x-int}}$

H.A. is not a joke! It's horizontal Asymptote

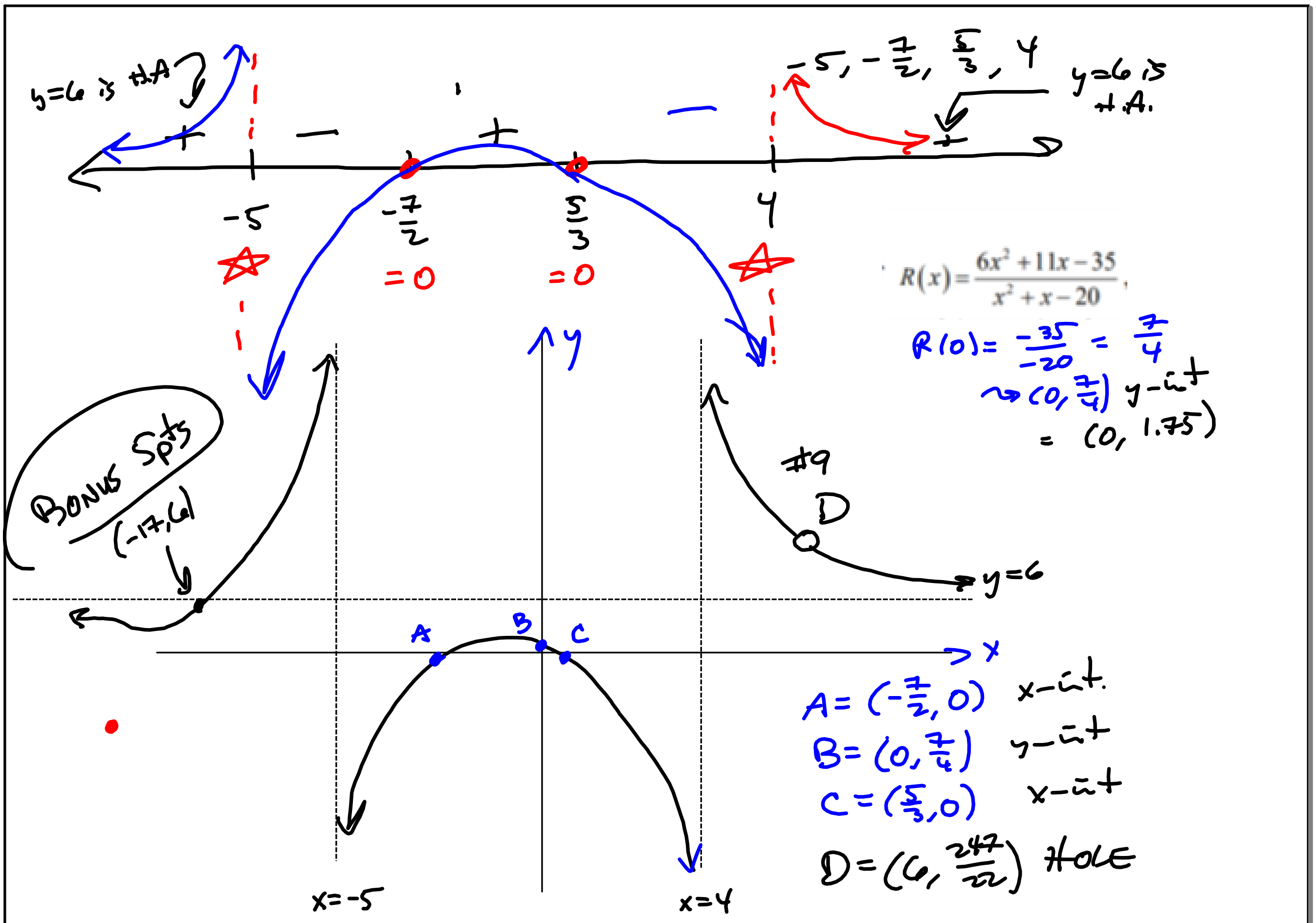
$|x|$ Big $\Rightarrow R(x) \approx \dfrac{6x^2}{1x^2} \therefore \boxed{6=y = \text{H.A.}}$

Sign Pattern is Key $<, \le, >, \ge$

Key values: $x=-5, 4, -\frac{7}{2}=-3.5, \frac{5}{3}=1.\overline{6}$



$R(x) = \dfrac{(2x+7)(3x-5)}{(x+5)(x-4)}$



No Graphing Calculator for an assist? Try Desmos and wolframalpha.com

https://www.wolframalpha.com/

Click on the little Earth

$$R(x)=\frac{6x^2+11x-35}{x^2+x-20} \overset{\text{SET}}{=} 6 = y = \text{H.A.}$$

Note: There *is* a subtle feature to this graph that I downplay on tests, but you should pick up on with a take-home, namely, the horizontal asymptote *does* intersect the graph of the function.

I'm willing to part with **5 bonus points** if you can find the point of intersection of $R(x)$ with its horizontal asymptote and label it with an ordered-pair label. I'm also looking for its effect on the graph. There's a little wiggle to this graph in the 1st quadrant.

$$\frac{6x^2+4x-35}{x^2+x-20} = 6\left(\frac{x^2+x-20}{x^2+x-20}\right)$$

$$\Rightarrow 6x^2+11x-35 = 6x^2+6x-120$$

$$5x = -85$$

$$x = \frac{-85}{5} = -17$$

$$\begin{array}{r} -120 \\ +\ 35 \\ \hline -85 \end{array}$$

Clearing fractions is also OK.

Want to keep muscles big for $\geq, <$, as well as $=$, & you can't clear fractions for $<, >$ (i.e., inequalities).

9. (2 pts) Sketch the graph of $Q(x)=\frac{6x^3-25x^2-101x+210}{x^3-5x^2-26x+120}$. All the work you did for #8 applies to this one, *except* for the *hole* in the graph of Q, which I expect you to find and clearly label in your graph.

$$R(x)=\frac{6x^2+11x-35}{x^2+x-20} = \frac{(2x+7)(3x-5)}{(x+5)(x-4)}$$

what I'm saying is

$$Q(x) = R(x)\cdot\frac{(x-c)}{(x-c)} = \frac{(2x+7)(3x-5)}{(x+5)(x-4)}\cdot\frac{(x-c)}{(x-c)}$$

The downstairs is "nicer"

$$x^3-5x^2-26x+120 = (x^2+x-20)(x-c)$$

Pull out the $x+5$ & $x-4$ to see $x-c$.

$$\begin{array}{r|rrrr} -5 & 1 & -5 & -26 & 120 \\ & & -5 & 50 & -120 \\ \hline 4 & 1 & -10 & 24 & 0 \\ & & 4 & -24 & \\ \hline & 1 & -6 & 0 & \end{array}$$

sweet! Sweeet!

$\to x-c = x-6$ HOLE @ $x=6$.

Double Check the numerator:

$$\begin{array}{r|rrrr} 6 & 6 & -25 & -101 & 210 \\ & & 36 & 66 & -210 \\ \hline & 6 & 11 & -35 & 0 \end{array}$$ ✓

$$\frac{(x-6)(6x^2+11x-35)}{(x-6)(x^2+x-20)} = Q(x)$$

$$\begin{array}{r} 36 \\ \times 6 \\ \hline 216 \end{array}$$

$= R(x)$ with a hole @ $x=6$!

$$R(6)\quad \frac{6(6)^2+11(6)-35}{6^2+6-20} = \frac{216+66-35}{36+6-20}$$

$$= \frac{247}{22} \to \text{HOLE}: \left(6, \frac{247}{22}\right)$$

10. (5 pts) Sketch the graph of $T(x)=\dfrac{6x^3-25x^2-101x+210}{3x^2+x-10}$, showing all intercepts and asymptotes. This was also built off #8, so use the zeros you found for the numerator in #8 to help you find the 3rd zero of this new numerator.

$3x^2+x-10$

$-30 = -(3)(2)(5)$

$3x^2+6x-5x-10$

$=3x(x+2)-5(x+2)$

$=(x+2)(3x-5)$

$T(x)=\dfrac{(2x+7)(3x-5)(x-6)}{(3x-5)(x+2)}$ by previous work.

Hole @ $x=\frac{5}{3}$!

Use $T^*(x)$ for $T(x)$ w/o the hole:

$T^*(x)=\dfrac{(2x+7)(x-6)}{x+2}$

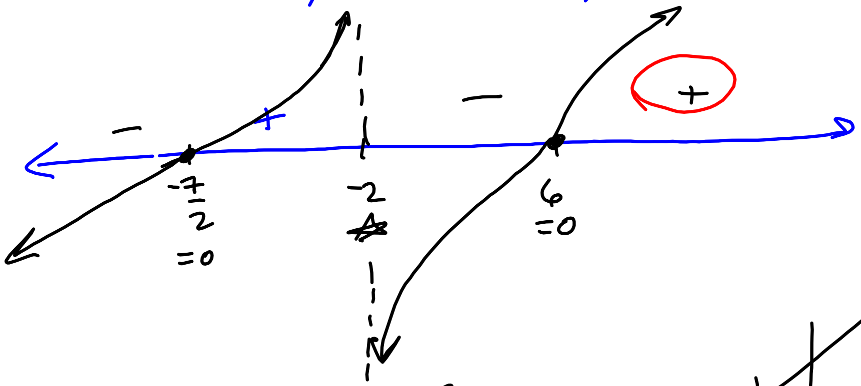
$D: \mathbb{R}\setminus\{-2, \frac{5}{3}\}$

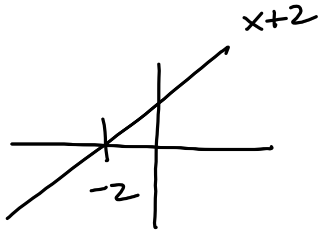
$\boxed{\text{V.A.}: x=-2}$ ($x=\frac{5}{3}$ is the hole)

$T^*(x)=0 \longrightarrow (2x+7)(x-6)=0 \longrightarrow x\in\{-\frac{7}{2}, 6\}$

$\boxed{x\text{-int}: (-\frac{7}{2},0),\ (6,0)}$ → Sign Pattern

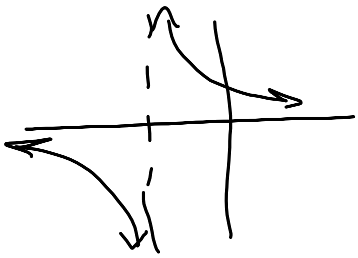
$x=-2, -\frac{7}{2}, 6 \rightsquigarrow x=-\frac{7}{2}, -2, 6$





$T^*(x)=\dfrac{(2x+7)(x-6)}{x+2}=\dfrac{2x^2+\cdots}{x+2}$

$x\longrightarrow \text{BIG}$ $\quad \dfrac{2x^2}{x}=2x$



$$\begin{array}{r} 2x-9 \\ x+2\,\overline{)\,2x^2-5x-42} \\ -(2x^2+4x) \\ \hline -9x-42 \end{array}$$

$\boxed{y=2x-9 \text{ is O.A.}}$

HOLE ② $x=\frac{5}{3}$, so $T^*(\frac{5}{3})=\dfrac{2(\frac{5}{3})^2-5(\frac{5}{3})-42}{(\frac{5}{3})+2}=\dfrac{2(\frac{25}{9})-\frac{25}{3}-42}{\frac{5+6}{3}}$

$=\dfrac{\frac{50}{9}-\frac{25}{3}\cdot\frac{3}{3}-\frac{42}{1}\cdot\frac{9}{9}}{\frac{11}{3}}=\dfrac{\frac{50-75-378}{9}}{\frac{11}{3}}$

$=\dfrac{-\frac{403}{9}}{\frac{11}{3}}=-\frac{403}{9}\cdot\frac{3}{11}=-\frac{403}{33}$

$\boxed{\text{HOLE ② } (\frac{5}{3}, -\frac{403}{33})}$

$42 \times 9 = 378$

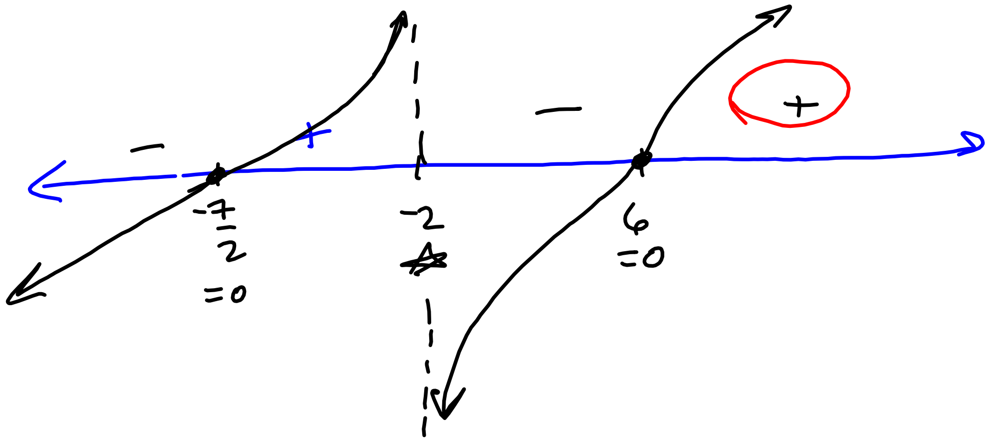
$378 + 75 = 453$

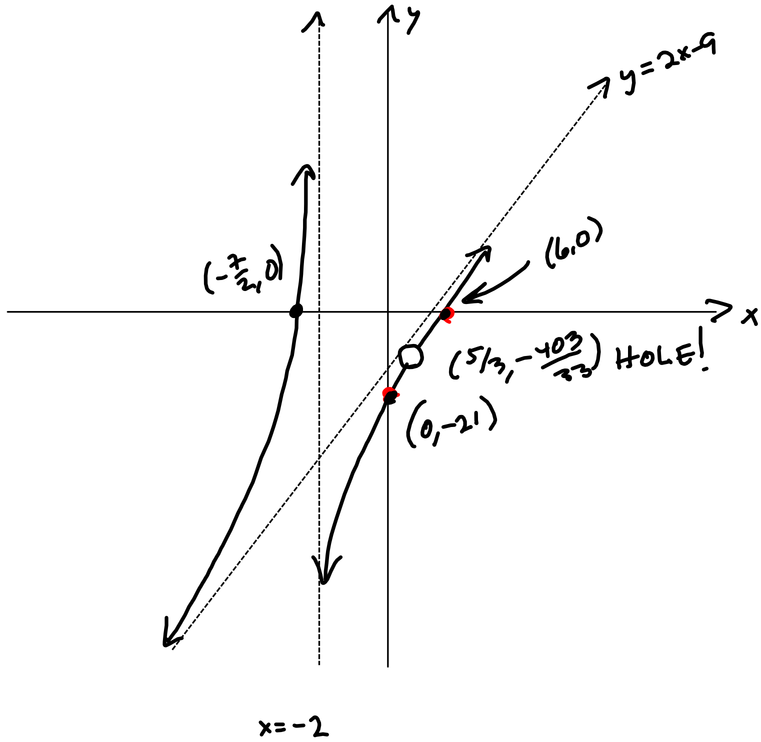
$453 - 50 = 403$

HOLE ② $\left(\frac{5}{3}, -\frac{403}{33}\right)$

V.A.: $x=-2$ ($x=\frac{5}{3}$ is the hole) ; O.A. $y=2x-9$

x-int: $(-\frac{7}{2},0)$, $(6,0)$; y-int: $(0,-21)$

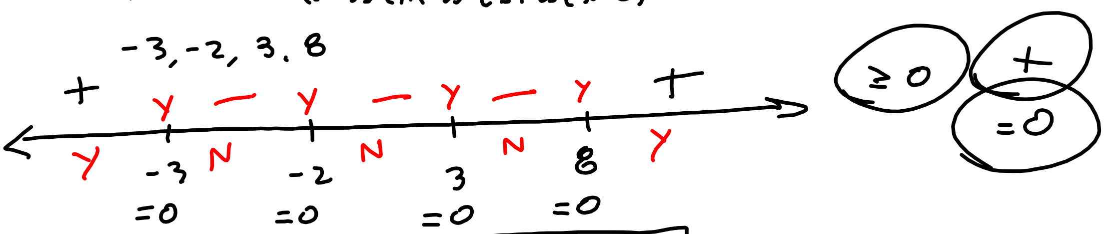




11. (2 pts) What is the domain of $W(x)=\sqrt{(x-3)^2(x+3)^3(x+2)^4(x-8)}$?

Domain: Need: $(x-3)^2(x+3)^3(x+2)^4(x-8) \geq 0$

$-3, -2, 3, 8$

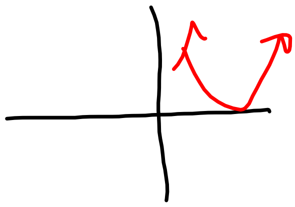


$= \boxed{(-\infty,-3] \cup \{-2,3\} \cup [8,\infty) = D(w)}$

$w(x) = \sqrt{f(x)} = \sqrt{x^{10} + \text{smaller degree}}$

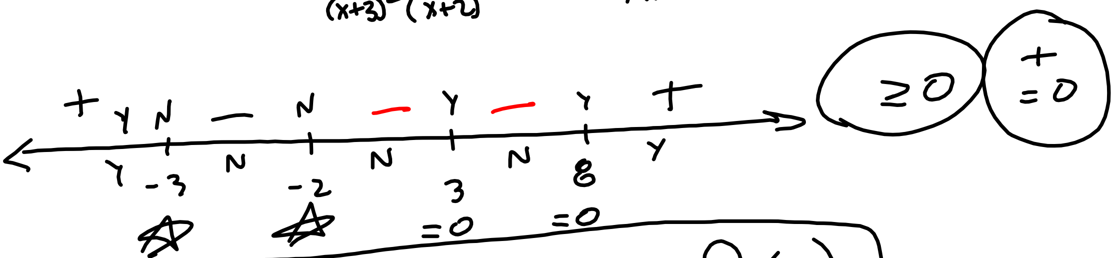
$\sqrt{\curvearrowleft \ldots \curvearrowright}$

$x^2,\ x^4,\ x^6,\ x^{20},\ x^{131313 2}$



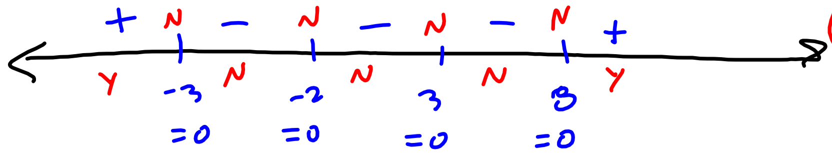
12. (2 pts) What is the domain of $K(x)=\sqrt{\dfrac{(x-3)^2(x-8)}{(x+3)^3(x+2)^4}}$?

Need $\dfrac{(x-3)^2(x-8)}{(x+3)^2(x+2)^4} \geq 0$ AND $(x+3)^3(x+2)^4 \neq 0$



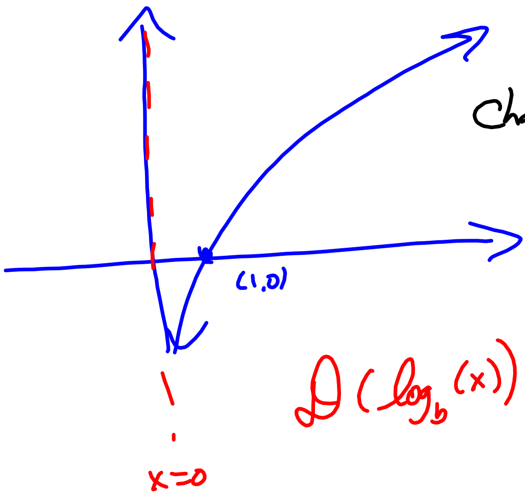
$= \boxed{(-\infty,-3) \cup \{3\} \cup [8,\infty) = D(K)}$

13. (Bonus) What is the domain of $J(x) = \log_3\left((x-3)^2(x+3)^3(x+2)^4(x-8)\right)$?



>0

$\mathcal{D}(J(x)) = (-\infty,-3) \cup (8,\infty)$



Chapter 4 Review.

$\mathcal{D}(\log_b(x)) = (0,\infty) \ \forall\, b>0.$