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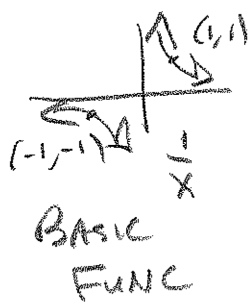
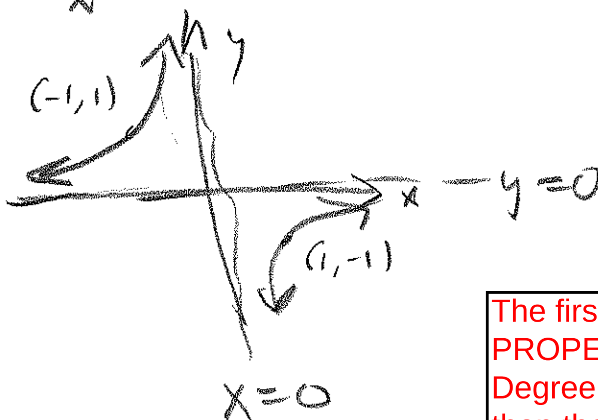
3.5 Graphs of Rational Functions and Rational Inequalities

Find all asymptotes, x- and y-intercepts, and sketch

1

$f(x) = -\frac{1}{x}$ $\mathcal{D} = \mathbb{R} \setminus \{0\}$

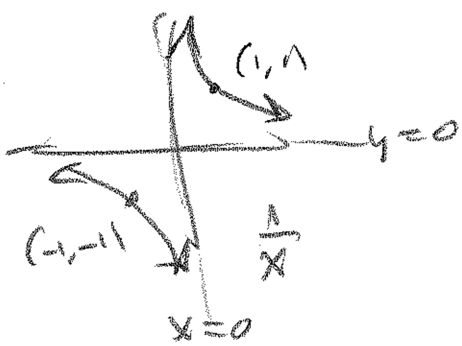


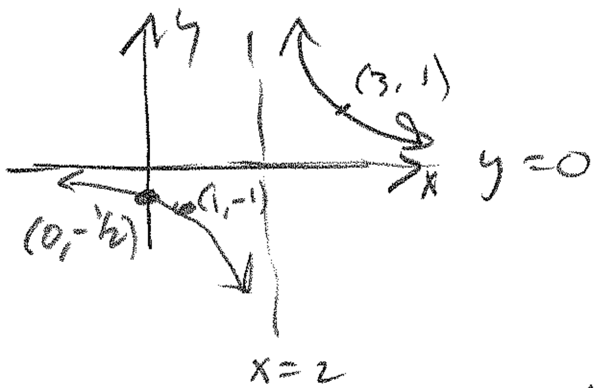


The first 4 exercises are PROPER rational functions: Degree downstairs is greater than the degree upstairs.

2

$f(x) = \frac{1}{x-2}$



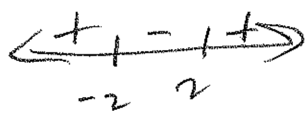


3

$f(x) = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$

y-int: $(0, f(0)) = \boxed{(0, -\frac{1}{4})}$

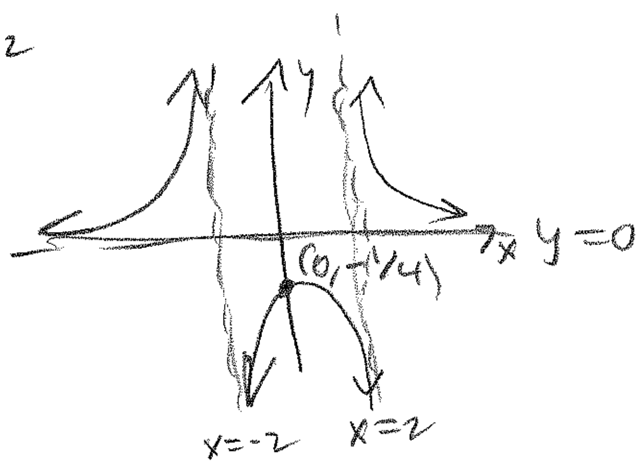
$\mathcal{D} = \mathbb{R} \setminus \{\pm 2\}$



$\boxed{\text{V.A.: } x = \pm 2}$

H.A.: $\frac{1}{x^2-4} \xrightarrow{x\to\infty} \frac{1}{\infty} = 0$

$\therefore$ $\boxed{y = 0 \text{ is H.A.}}$



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3.5 Graphs of Rational Functions and Rational Inequalities

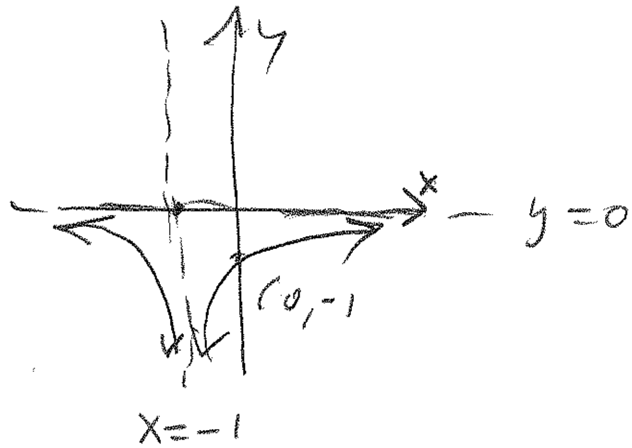
4

$f(x) = -\frac{1}{(x+1)^2}$

$\mathcal{D} = \mathbb{R} \setminus \{-1\}$

V.A.: $x = -1$

H.A.: $y = 0$ (f is "proper")



$\overset{-}{\longleftarrow} \underset{-1}{+} \overset{-}{\longrightarrow}$

$f(0) = -1 \rightsquigarrow (0, -1)$

When the degree is the same, up and down, you have a horizontal asymptote (H.A.)

5

$f(x) = \frac{x-3}{x+2}$

This is the first IMPROPER rational function. They're IMPROPER when the degree upstairs is the same or greater than the degree downstairs.

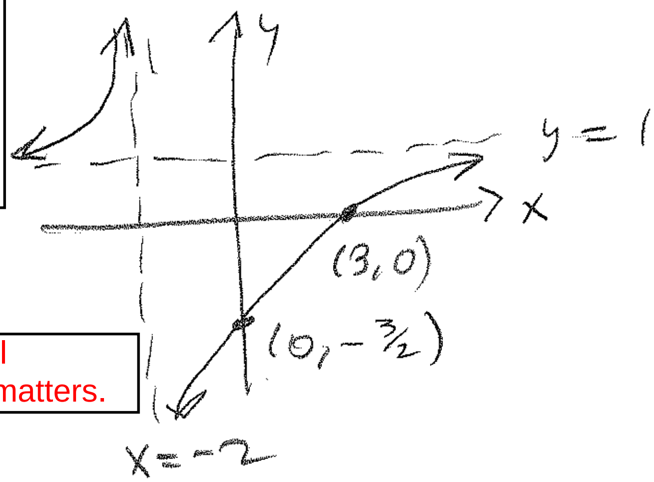
$\mathcal{D} = \mathbb{R} \setminus \{-2\}$

V.A.: $x = -2$

H.A.: $\frac{x-3}{x+2} \xrightarrow{x \to BIG} \frac{x}{x} = 1$

$y = 1$

Above is the work for the horizontal asymptote. Only the biggest stuff matters.



$x - 3 = 0$

$x = 3 \rightsquigarrow \boxed{(3, 0)}$

$\overset{+}{\longleftarrow} \underset{-2}{|} \overset{-}{\;} \underset{3}{|} \overset{+}{\longrightarrow}$ Sign Pattern informs the graph.

$f(0) = \frac{-3}{2}$

Horizontal asymptote is great for sign pattern. It tells us the end behavior. Here, y = 1 is the H.A., and this tells us that the function is positive on the far right and far left. USE this concept! Otherwise, things just move too slow and are too complicated.

3.5 Graphs of Rational Functions and Rational Inequalities

6) $f(x) = \frac{2x+1}{x-1}$

Any time degree of numerator is same as degree of denominator, just look @ highest degree terms for a horizontal asymptote.

$\frac{2x}{x} = 2 \Longrightarrow \boxed{y = 2 \text{ is H.A.}}$

$\mathcal{D} = \mathbb{R} \setminus \{1\}$

$\boxed{x = 1 \text{ is V.A.}}$

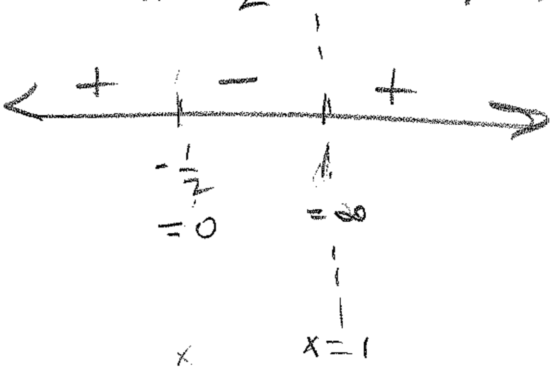
$f(0) = \frac{1}{-1} = -1 \rightsquigarrow \boxed{(0,-1)}$

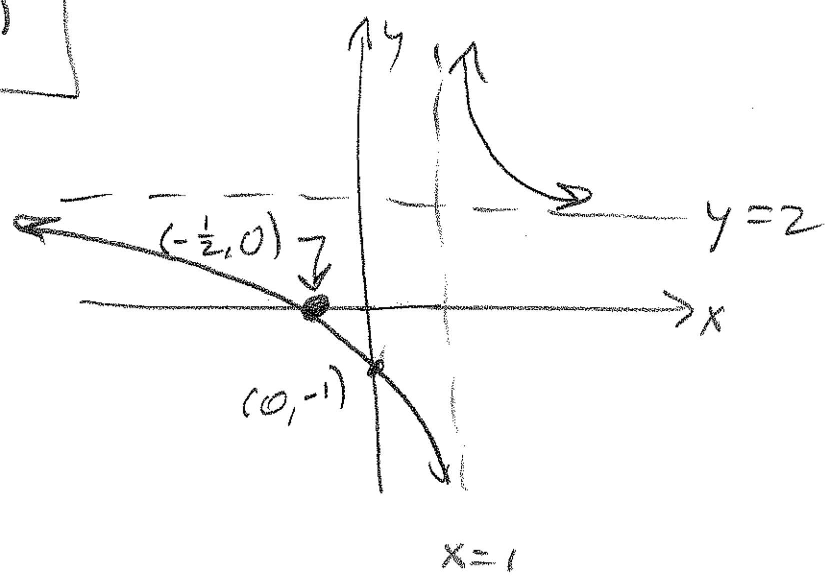
$\frac{2x+1}{x-1} = 0 \Longrightarrow$

$2x+1 = 0 \Longrightarrow$

$2x = -1 \Longrightarrow$

$x = -\frac{1}{2} \rightsquigarrow (-\frac{1}{2}, 0)$





7

$f(x) = \frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$

$\mathcal{D} = \mathbb{R} \setminus \{\pm 1\}$

$\boxed{x = \pm 1 \text{ V.A.}}$

Degree of denominator is greater than degree of numerator $\Rightarrow$ "PROPER" $\Rightarrow$ $\boxed{y=0 \text{ is H.A.}}$

$f(x) = 0$

$\frac{x}{x^2-1} = 0$

$\boxed{x=0} \rightsquigarrow (0,0)$

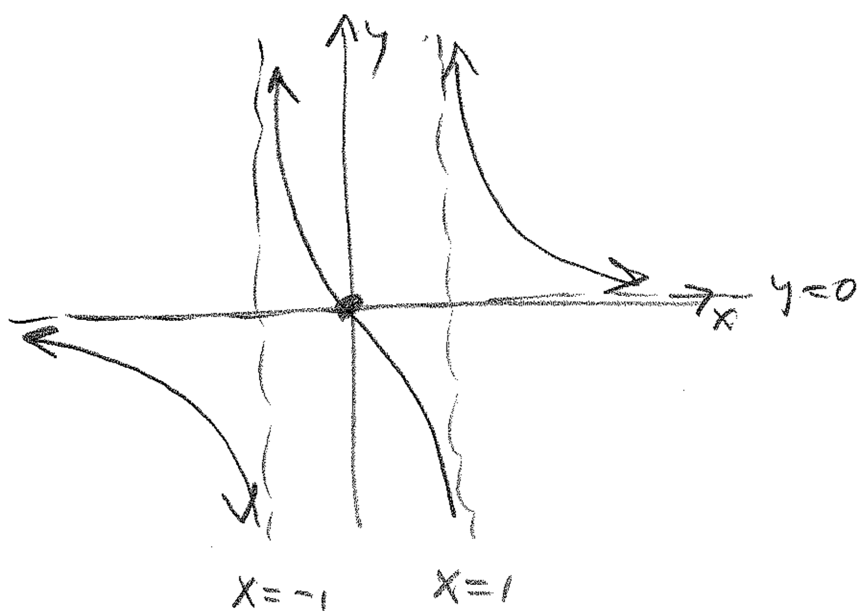
$-$, $+$, $-$, $+$

$-1 \quad 0 \quad 1$

Might Test $x=2$ to be sure the "+" on the right is correct.

$\frac{2}{2^2-1} = \frac{2}{3} > 0$ "+"

Yep.



3.5 Graphs of Rational Functions and Rational Inequalities

8

$$f(x) = \frac{4x}{x^2 - 2x + 1} = \frac{4x}{(x-1)^2}$$

$\mathcal{D} = \mathbb{R} \setminus \{1\}$

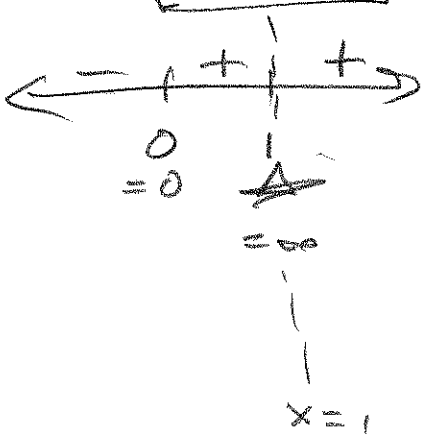
$\boxed{\text{V.A. is } x = 1}$

$\boxed{\text{H.A. is } y = 0}$

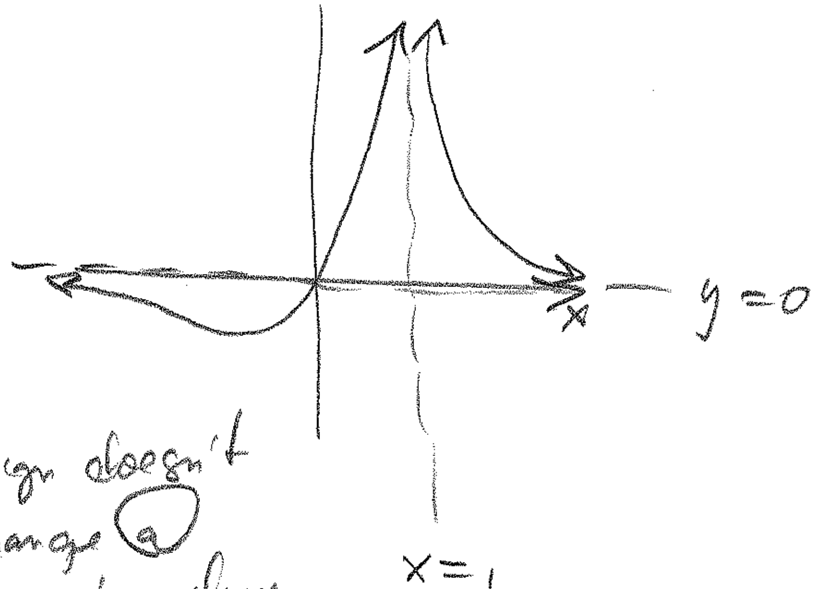
(f is "PROPER.")

$f(0) = \frac{0}{\ldots} = 0$.

$\rightsquigarrow \boxed{(0, 0)}$



Sign doesn't change @ $x = 1$ boundary, because $x = 1$ is root of $m = 2$



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3.5 Graphs of Rational Functions and Rational Inequalities

9

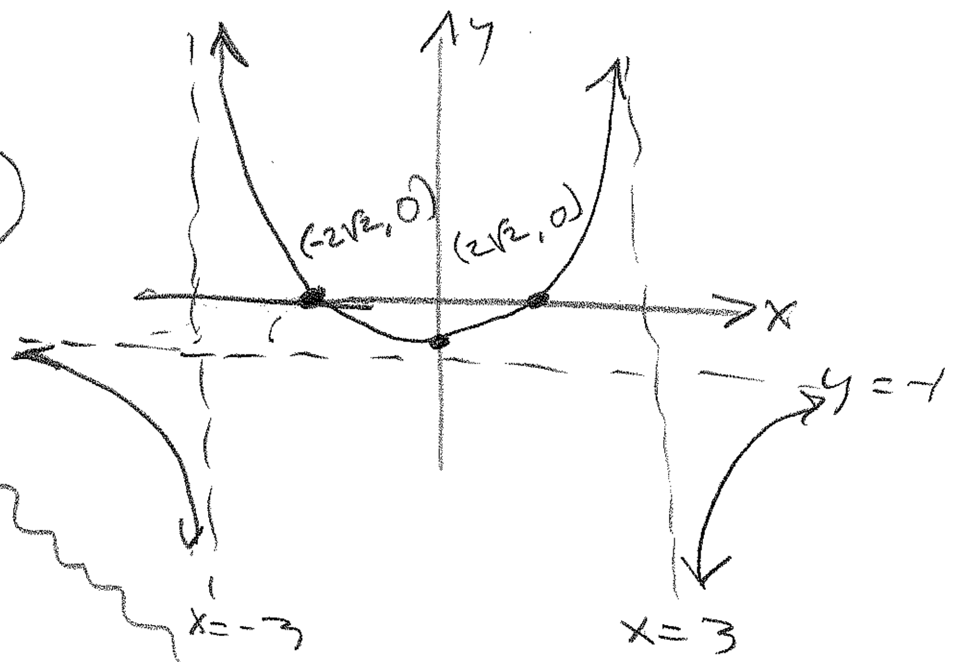
$$f(x) = \frac{8-x^2}{x^2-9} = \frac{-x^2+8}{x^2-9} = \frac{-(x^2-8)}{x^2-9} = \frac{-(x-2\sqrt{2})(x+2\sqrt{2})}{(x-3)(x+3)}$$

$\mathcal{D} = \mathbb{R} \setminus \{\pm 3\}$

$\boxed{\text{VA: } x = \pm 3}$

H.A.: $\frac{-x^2}{x^2} = -1$

$y = -1$ is EB



$$\frac{8-x^2}{x^2-9} = 0$$

$8 - x^2 = 0$

$-x^2 = -8$

$x^2 = 8$

$x = \pm\sqrt{8}$

$x = \pm 2\sqrt{2}$

$\boxed{(2\sqrt{2}, 0),\ (-2\sqrt{2}, 0)}$

$f(0) = \frac{8}{-9} \rightsquigarrow (0, -\frac{8}{9})$

$$\overset{-}{}\ -3\ \overset{+}{}\ -2\sqrt{2}\ \overset{-}{}\ 2\sqrt{2}\ \overset{+}{}\ 3\ \overset{-}{}$$

E..

3.5 Graphs of Rational Functions and Rational Inequalities

10

$f(x) = \dfrac{2x^2+8x+2}{x^2+2x+1} = \dfrac{2(x^2+4x+2)}{(x+1)^2}$

$\mathcal{D} = \mathbb{R} \setminus \{-1\}$

$\boxed{\text{V.A. is } x=-1}$

H.A.: $\dfrac{2x^2}{x^2} = 2$

$\boxed{y=2 \text{ is H.A.}}$

$f(0) = \dfrac{2}{1} = 2 \rightsquigarrow \boxed{(0,2)}$

$f(x) = 0$:

$\dfrac{2x^2+8x+2}{(x+1)^2} = 0$

$2x^2+8x+2 = 0$

$x^2+4x+1 = 0$

$a=1, b=4, c=1$

$b^2-4ac = 4^2-4(1)(1)$
$= 16-4$
$= 12$

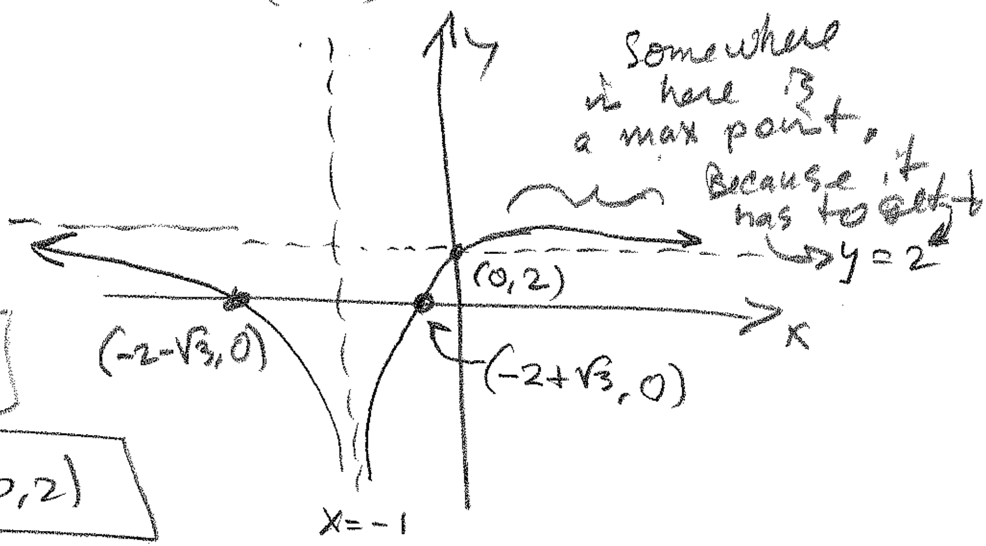
$\sqrt{12} = 2\sqrt{3}$

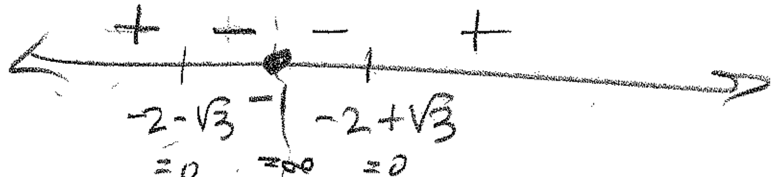
$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-4 \pm 2\sqrt{3}}{2}$

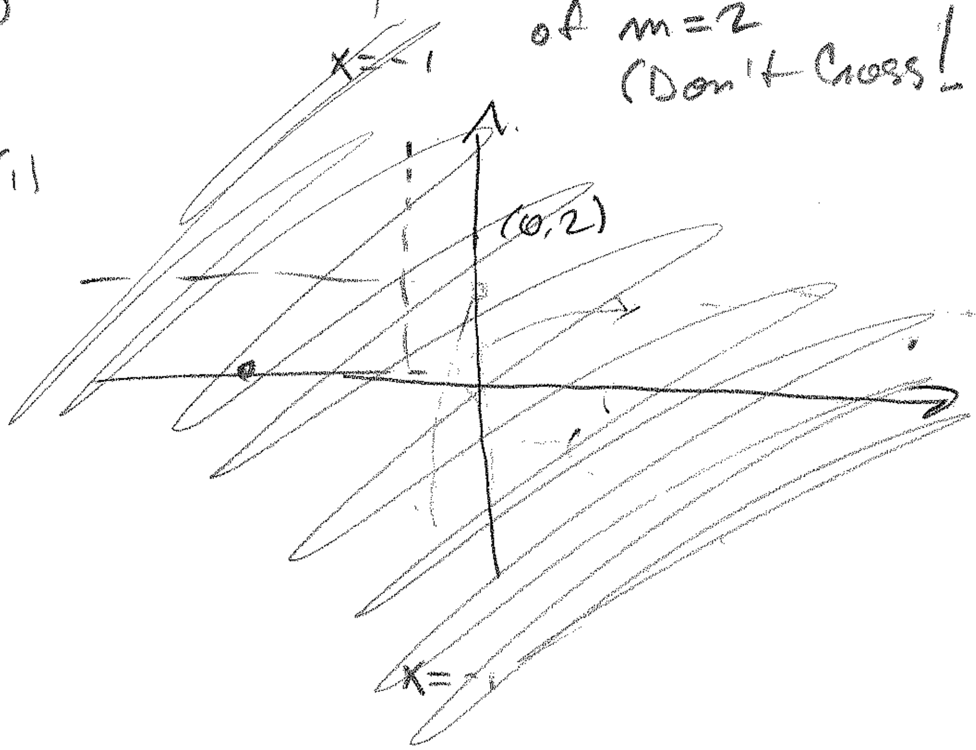
$= -2 \pm \sqrt{3}$

$\rightsquigarrow (-2+\sqrt{3}, 0), (-2-\sqrt{3}, 0)$





Almost missed
x = -1 is root
of m = 2
(Don't Cross!)



Find oblique asymptote & sketch the graph of the fn.

11

$$f(x) = \frac{x^2+1}{x}$$

Degree of numerator is GREATER than that of denominator, Use Division to find oblique Asymptote

$\mathcal{D} = \mathbb{R} \setminus \{0\}$

$\boxed{\text{V.A. is } x = 0}$

H.A.: NONE!

O.A.

① O.A.

$$\begin{array}{r} x + r\,1 \\ x \overline{) x^2 + 0x + 1} \\ -(x^2 \qquad\quad) \\ \hline 1 \end{array}$$

$f(x) = x + \frac{1}{x}$

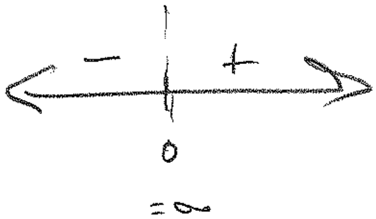
② $\frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$ → O.A.

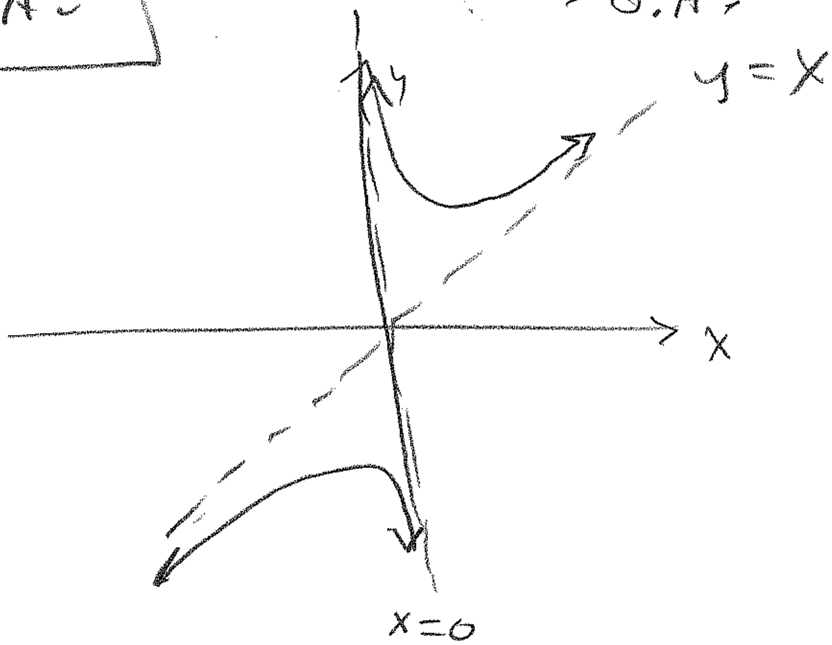
③
$$\begin{array}{c|ccc} 0 & 1 & 0 & 1 \\ & & 0 & 0 \\ \hline & 1 & 0 & 1 \\ & x & c & r \end{array}$$

$\frac{x^2+1}{x} = x + \frac{1}{x}$ → O.A.

UPSHOT:

$\boxed{y = x \text{ is O.A.}}$





MAPLE / GRAPHING Calculator say

$f(x)=0$ when

$x \approx 2.391382381, -2.164247938, 0.7728655577$

3 zeros, degree 3 $(-x^3+x^2+5x-4) \Longrightarrow$

All 3 are multiplicity $m=1$.

$$f(x) \approx \frac{(x-2.39138)(x+2.16425)(x-.77287)}{(x+2)(x-1)}$$

For the oblique asymptote:

$$\begin{array}{r} -x+2 \quad r\ x \\ x^2+x-2 \overline{\big)\, -x^3+x^2+5x-4} \\ -(-x^3-x^2+2x) \\ \hline 2x^2+3x-4 \\ -(2x^2+2x-4) \\ \hline x \end{array}$$

Critical:

$-2.16, -2, .77, 1, 2.39$

$\longleftarrow -2.16 \quad -2 \quad .77 \quad 1 \quad 2.39 \longrightarrow$

0

So $$\frac{-x^3+x^2+5x-4}{x^2+x-2} = -x+2 + \frac{x}{x^2+x-2}$$

Tells us EB

O.A. is $\boxed{y=-x+2}$

$f(0) = \frac{-4}{-2} = 2$

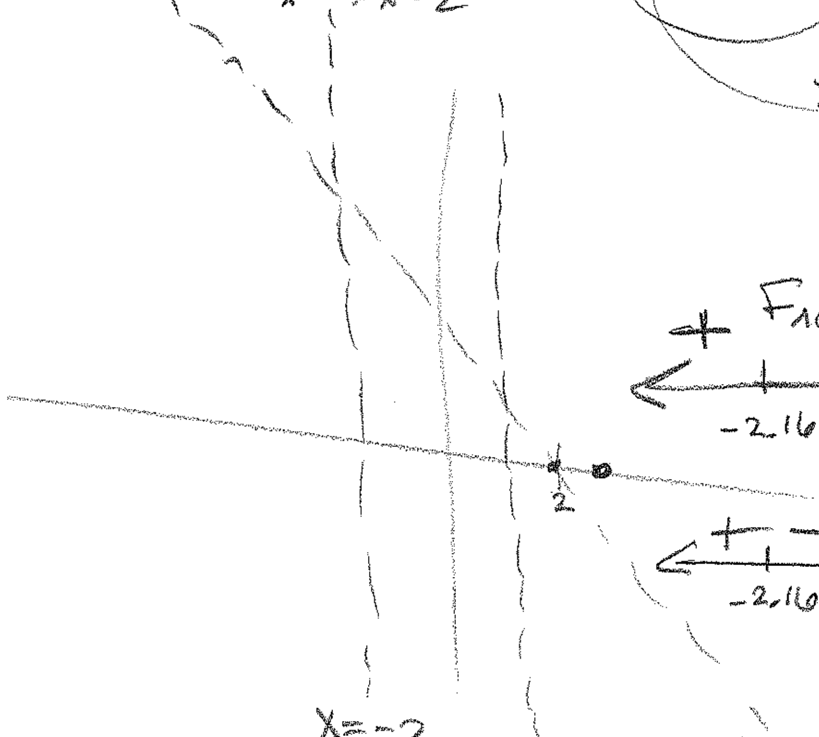
From EB:

$+ \qquad\qquad -$

$\longleftarrow -2.16 \quad -2 \quad .77 \quad 1 \quad 2.39 \longrightarrow$

$+ \quad - \quad + \quad - \quad + \quad -$

$\longleftarrow -2.16 \quad -2 \quad .77 \quad 1 \quad 2.39 \longrightarrow$



$x=-2 \qquad x=1 \qquad y=-x+2$

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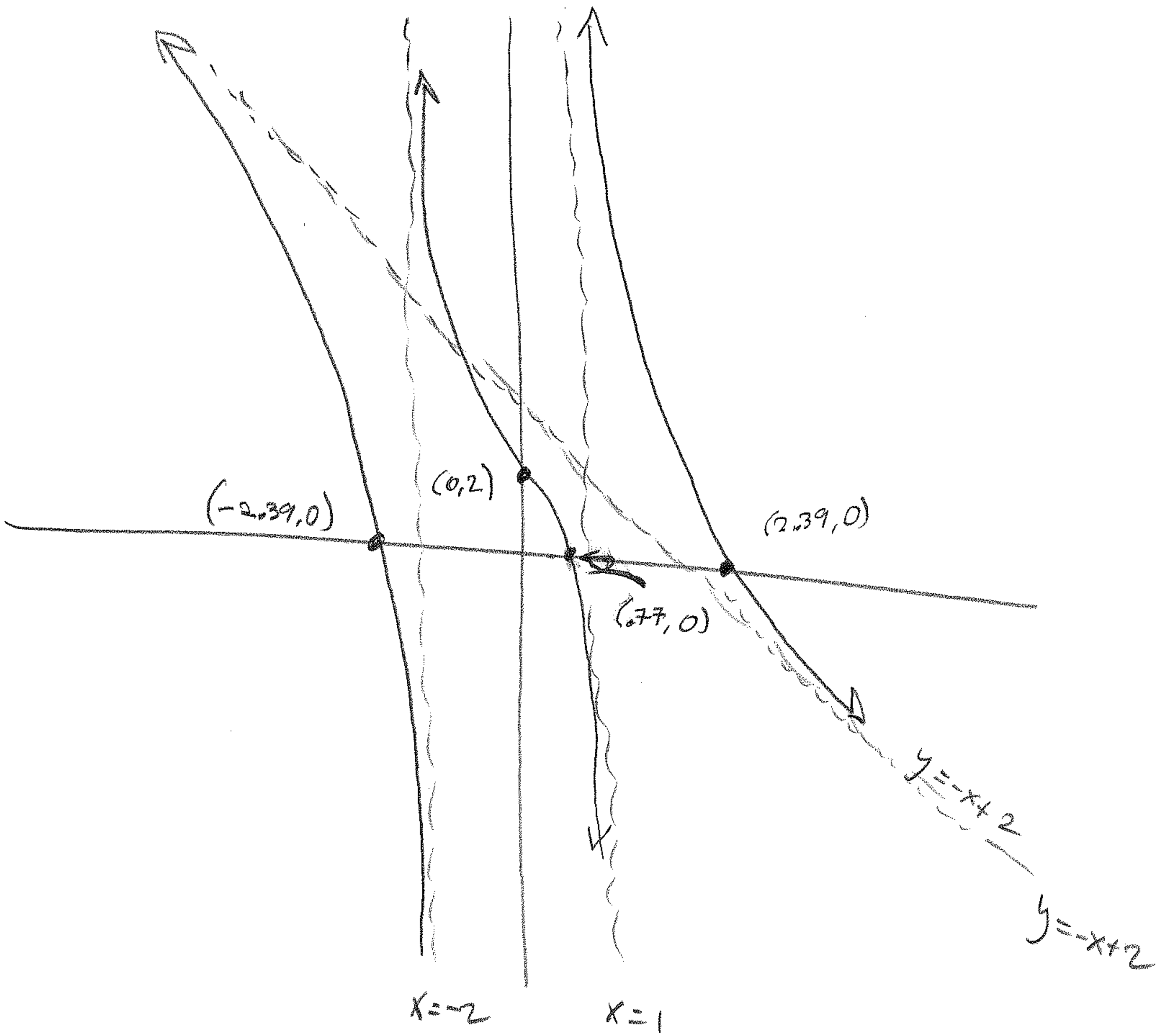
3.5 Graphs of Rational Functions and Rational Inequalities



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3.5 Graphs of Rational Functions and Rational Inequalities

12

$$w > \frac{w-5}{w-3}$$

$$w - \frac{w-5}{w-3} > 0$$

$$\left(\frac{w}{1}\right)\left(\frac{w-3}{w-3}\right) - \frac{w-5}{w-3} > 0$$

$$\frac{w^2-3w-(w-5)}{w-3} > 0$$

$$\frac{w^2-3w-w+5}{w-3} > 0$$

$$\frac{w^2-4w+5}{w-3} > 0$$

$w^2-4w+5=0$:

$a=1, b=-4, c=5$

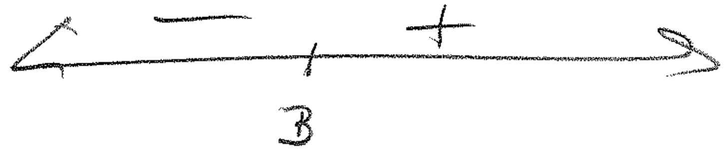
$b^2-4(a)(c)$

$=(-4)^2-4(1)(5)$

$=16-20=-4$

No real roots in numerator.

$w=3$ is only critical #



$$\boxed{w \in (3, \infty)}$$

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3.5 Graphs of Rational Functions and Rational Inequalities

#s 95 – 114 solve the inequality

13

$$\frac{x-4}{x+2} \le 0 \qquad \boxed{x \in (-2,4]}$$

$+ \quad | \quad - \quad | \quad +$

$-2 \qquad 4$

$2 \notin D$

$f(0) = \frac{-4}{2} = -2 < 0$ " — "

14

$$\frac{g-2}{g+3} < 2$$

$$\frac{g-2}{g+3} - 2 < 0$$

$$\frac{g-2-2(g+3)}{g+3} < 0$$

$$\frac{g-2-2g-6}{g+3} < 0$$

$$\frac{-g-8}{g+3} < 0$$

$f(0) = -\frac{8}{3}$ " — "

$| \quad | \quad -$

$-8 \qquad -3$

from g-ist finishe

$- \quad | \quad + \quad | \quad -$

$-8 \qquad -3$

$$\Rightarrow \boxed{x \in (-\infty,-8) \cup (-3,\infty)}$$

15

$$\frac{w^2-w-6}{w-6} \ge 0$$

$$\frac{(w-3)(w+2)}{w-6} \ge 0$$

$- \quad | \quad + \quad | \quad - \quad | \quad +$

$-2 \qquad 3 \qquad 6$

$$\Rightarrow \boxed{x \in [-2,3] \cup (6,\infty)}$$

$6 \notin D$

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3.5 Graphs of Rational Functions and Rational Inequalities

16

$$\frac{1}{x+2} > \frac{1}{x-3}$$

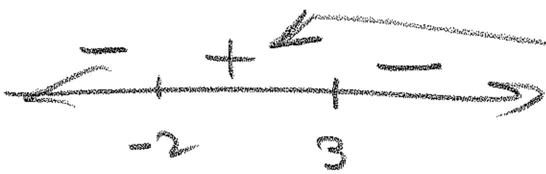
$$\frac{1}{x+2} - \frac{1}{x+3} > 0$$

$$\left(\frac{1}{x+2}\right)\left(\frac{x-3}{x-3}\right) - \left(\frac{1}{x-3}\right)\left(\frac{x+2}{x+2}\right) > 0$$

$$\frac{x-3-(x+2)}{(x+2)(x-3)} > 0$$

$$\frac{x-3-x-2}{(x+2)(x-3)} > 0$$

$$\frac{-5}{(x+2)(x-3)} > 0$$



$$f(0) = \frac{-5}{(2)(-3)} = \frac{-5}{-6} = \frac{5}{6} \quad \text{" + "}$$

$$\boxed{x \in (-2, 3)}$$

17

$$x < \frac{3x-8}{5-x}$$

$$(x)\left(\frac{5-x}{5-x}\right) - \frac{3x-8}{5-x} < 0$$

$$\frac{5x - x^2 - 3x + 8}{5-x} < 0$$

$$\frac{(-1)}{(-1)}\left(\frac{-x^2+2x+8}{5-x}\right) < 0$$

$$\frac{x^2-2x-8}{x-5} < 0$$

$$\frac{(x-4)(x+2)}{x-5} < 0$$

$$f(0) = \frac{-8}{-5} = \frac{8}{5} \quad \text{" + "}$$

$$\boxed{x \in (-\infty, -2) \cup (4, 5)}$$

These inequalities are grist for graphing exercise.
The earlier graphs are grist for exercise on inequalities.
Go ahead and build questions off these questions, until you are confident of your mastery.

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3.5 Graphs of Rational Functions and Rational Inequalities

18

$$\frac{(x-3)(x+1)}{x-5} \geq 0$$

$$f(0) = \frac{(-3)(1)}{(-5)} \quad \text{"}+\text{"}$$

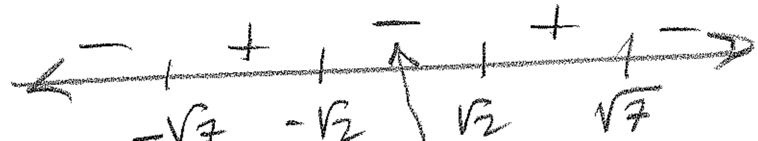
$$\boxed{x \in [-1,3] \cup (5,\infty)}$$

$5 \notin \mathcal{D}$

19

$$\frac{x^2-7}{2-x^2} \geq 0$$

$$\frac{(x-\sqrt{7})(x+\sqrt{7})}{(\sqrt{2}-x)(\sqrt{2}+x)} \geq 0$$



$$f(0) = -\frac{7}{2} \quad \text{"}-\text{"}$$

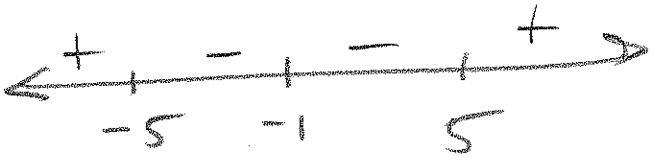
$$\boxed{x \in [-\sqrt{7},-\sqrt{2}) \cup (\sqrt{2},\sqrt{7}]}$$

$\pm\sqrt{2} \notin \mathcal{D}$

20

$$\frac{x^2+2x+1}{x^2-2x-15} \geq 0$$

$$\frac{(x+1)^2}{(x-5)(x+5)} \geq 0$$



$x=-1$ has $m=2$ (Don't change sign.)

$x \in (-\infty,-5) \cup \{1\}$
$\cup (5,\infty)$

$x=5 \notin \mathcal{D}$

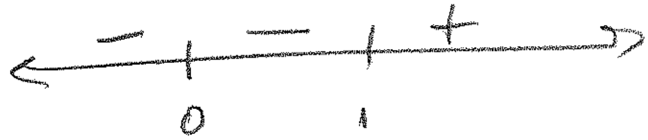
$x=1$ is one spot where "$=0$" is satisfied for ≥ 0!

21

$$\frac{1}{w} > \frac{1}{w^2}$$

$$\frac{1}{w}\cdot\frac{w}{w} - \frac{1}{w^2} > 0$$

$$\frac{w-1}{w^2} > 0$$



$$\boxed{w \in (1,\infty)}$$