Geometric Sequences & Series

$a_n = ar^{n-1}$ is the n^{th} term of a geometric sequence.

r = common ratio

a = the 1^{st} term

$a = 1, r = \frac{1}{2}$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots,$

Is this geometric? Yes.

$\frac{a_{n+1}}{a_n} = r$

$\frac{a_2}{a_1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

$a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1}$

$\frac{a_3}{a_2} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$

$\frac{a_4}{a_3} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{8} \cdot \frac{4}{1} = \frac{1}{2}$

$3, \frac{3}{4}, \frac{3}{16}, \frac{3}{64}, \dots$

$a = 3$

$r = \frac{1}{4}$

$a_1 = 3 \cdot \left(\frac{1}{4}\right)^{1-1} = 3\left(\frac{1}{4}\right)^0 = 3$ ✓

$a_2 = 3\left(\frac{1}{4}\right)^{2-1} = 3\left(\frac{1}{4}\right) = \frac{3}{4}$

$a_n = 3 \cdot \left(\frac{1}{4}\right)^{n-1}$

Apr 22-8:59 AM

$\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \ldots$

$a = \frac{1}{16}$ $\boxed{a_n = \frac{1}{16} \cdot \left(\frac{1}{2}\right)^{n-1}}$

$r = \frac{1}{2}$

$S_n = \sum_{k=1}^{n} a_k$ = "n^{th} (partial) sum" =

$= a_1 + a_2 + a_3 + \cdots + a_n$

$a_k = 2k$

$S_5 = \sum_{k=1}^{5} a_k = \sum_{k=1}^{5} (2k) = 2(1) + 2(2) + 2(3) + 2(4) + 2(5)$

$= 2 + 4 + 6 + 8 + 10$

$= 30$

Calc Students observe: $2(1) + 2(2) + 2(3) + 2(4) + 2(5)$

$= 2[1 + 2 + 3 + 4 + 5]$

$= 2[15] = 30$

So $\sum (2k) = 2 \sum k$ $\cancel{\int 2x\,dx = 2\int x\,dx}$

Claim: $\displaystyle\sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right)$

Proof:

$$S_n = \sum_{k=1}^{n} a\cdot r^{k-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Multiply by r:

$$S_n r = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

observe the telescope:

$$\begin{aligned} S_n - S_n r = {} & a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ & -(\quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \end{aligned}$$

$$S_n(1-r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r} = a\left(\frac{1-r^n}{1-r}\right) \blacksquare$$

EXAMPLE: $\sum_{k=1}^{5} 3\cdot 2^{k-1} = 3\left(\frac{1-2^5}{1-2}\right) = 3\left(\frac{-31}{-1}\right) = 93$

$3\left[2^0 + 2^1 + 2^2 + 2^3 + 2^4\right]$

$= 3\left[1 + 2 + 4 + 8 + 16\right] = 3\left[31\right] = 93$

Recall, Savings account with compound interest.

$$A = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

Example: \$500, compounded weekly, is deposited, and draws 7% interest for 12 years. what's the future value

$P = \$500$
$r = .07$
$m = 52$
$t = 12$

I'd use the 2^{nd} version, with the m's in it for better precision.
Students tend to lose points by rounding-off $i = \frac{r}{m} = \frac{.07}{52}$ owie!

$$A = P\left(1+\frac{r}{m}\right)^{mt} = 500\left(1+\frac{.07}{52}\right)^{52 \cdot 12}$$

$$\approx \$1157.53$$

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500*(1+.07/52)^(
12*52)
     1157.529441
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Apr 22-9:28 AM

An Annuity is a sequence of periodic payments into a bank account drawing interest.

Interest is compounded at the end of each period.

Payments are made at the end of each period.

Monthly $\overbrace{\text{payments}}^{\text{of \$300}}$ are made into an account earning 7% APR compounded monthly for 18 yrs. How much is in the bank at the end of the 18 yrs?

The general case

n payments over n periods, periodic rate i

R = payment amount.

1st pmt : $A = R(1+i)^{n-1}$

2nd pmt : $A = R(1+i)^{n-2}$

3rd pmt : $A = R(1+i)^{n-3}$

$\vdots$

nth pmt : $A = R(1+i)^{n-n} = R$

Add 'em up! :

$$FV = R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1}$$

$$\sum_{k=1}^{n} ar^{k-1} = a\left(\frac{1-r^n}{1-r}\right) = R\left(\frac{1-(1+i)^n}{1-(1+i)}\right) = R\left(\frac{1-(1+i)^n}{i}\right)$$

$1-(1+i) = 1-1-i = -i$ idiot.

Monthly payments of \$300 are made into an account earning 7% APR compounded monthly for 18 yrs. How much is in the bank at the end of the 18 yrs?

$$FV = R\left[\frac{1-\left(1+\frac{r}{m}\right)^{mt}}{-\left(\frac{r}{m}\right)}\right]$$

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500*(1+.07/52)^(
12*52)
      1157.529441
300*((1-(1+.07/1
2)^(12*18))/(.07
/12))
     -129216.308
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Monday, the 2nd @ 9:10.

is off by a sign, Steve.

$$FV = R\left[\frac{(1+r/m)^{mt}-1}{\left(\frac{r}{m}\right)}\right] \approx \boxed{\$129{,}216.31}$$