Long Division

$\frac{22}{3} = 7 \ldots$

$\frac{15}{3} = 5 \ldots$

$$\begin{array}{r} 750 \text{ r } 1 \\ 3 \overline{)2251} \\ -2100 \\ \hline 151 \\ -150 \\ \hline 1 \end{array}$$

1 is less than 3. Done

This says

$2251 = 3(750) + 1$

OR

$\frac{2251}{3} = 750 + \frac{1}{3}$

Divide $x^4 - 2x^2 + 5x - 1$ by $x^2 + 5$

Synthetic Division doesn't work, b/c

$\frac{x^4}{x^2} = x^2$

$\frac{-3x^2}{x^2} = -3$

$$\begin{array}{r} x^2 - 3 \quad r \; 5x+14 \\ x^2+5 \overline{) x^4 + 0x^3 + 2x^2 + 5x - 1} \\ -(x^4 \qquad + 5x^2) \\ \hline 0 \quad 0 \quad -3x^2 + 5x - 1 \\ -(-3x^2 \qquad -15) \\ \hline 5x + 14 \end{array}$$

$5x + 14$ = Remainder

5x is lower degree than x^2+5.

DONE.

This says

$x^4 + 2x^2 + 5x - 1 = (x^2+5)(x^2-3) + 5x + 21$

Dividend = Divisor · Quotient + Remainder

OR

$\frac{x^4 + 2x^2 + 5x - 1}{x^2 + 5} = x^2 - 3 + \frac{5x+21}{x^2+5}$

$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \frac{\text{Remainder}}{\text{Divisor}}$

Test question:

A polynomial with real coefficients has zeros x = -1, 2, -5 and 2 - 3*i*. Give me its factored form, baby!

ANS: $(x-(-1))(x-2)(x-(-5))(x-(2-3i))(x-(2+3i))$

$= (x+1)(x-2)(x+5)(x-(2-3i))(x-(2+3i))$

Conjugate Pairs Theorem. To NOT have imaginary stuff when we expand, we need zeros that aren't real to come in conjugate pairs.

$z = 2-3i$ has conjugate $\bar{z} = 2+3i$

$f1 := x \to (x+1)\cdot(x-2)\cdot(x+5)\cdot(x-(2-3\cdot I))$

$x \to (x+1)\ (x-2)\ (x+5)\ (x-2+3\,I)$

$expand(\,f1(x)\,)$

$x^4 + 2x^3 - 15x^2 + 4x + 3\,I x^3 + 12\,I x^2 - 21\,I x + 20 - 30\,I$

ugh! Yuck!

See how there're *i*'s in the above? We want *reals* ONLY!!!!
Check this out:

$f2 := x \to f1(x)\cdot(x-(2+3\cdot I))$ $= (x+1)(x-2)(x+5)(x-(2-3i))(x-(2+3i))$

$x \to f1(x)\ (x-2-3\,I)$

$expand(\,f2(x)\,)$

$x^5 - 10x^3 + 70x^2 - 51x - 130$

Sweet!

See? The conjugate pair cleared out all the imaginary crap.

So when we say "Real Coefficients"
You should be thinking Conjugate Pairs
for any non-real roots.

zeros of f(x).

Rational Zeros!

$(2x+3)(5x-7) = 10x^2+x-21$

SET 0 $\Rightarrow$ $2x+3=0$ OR $5x-7=0$

$2x=-3$ $\quad$ $5x=7$

$x=-\frac{3}{2}$ $\quad$ $x=\frac{7}{5}$

$x=-\frac{3}{2}$ is a root (zero)

NOTE: 3 is a factor of 21.

and 2 " " " " 10.

It always works that way.

$x^5 - 10x^3 + 70x^2 - 51x - 130$

Any fraction that makes this thing zero must be one of the following:

$\frac{p}{q} = \frac{\text{factor of } 130}{\text{factor of } 1}$

(130 = last term
1 = leading coefficient)

$$\begin{array}{r|l} 2 & 130 \\ 5 & 65 \\ & 13 \end{array}$$

All possible guesses.

$\frac{p}{q}$: $\pm 1, \pm 2, \pm 5, \pm 10, \pm 13$

Possible Rational zeros of

$f(x) = 2x^3+5x-15$

$\frac{p}{q}$: $\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}$

All the possibles

$x^5 - 10x^3 + 70x^2 - 51x - 130$

$\frac{p}{q}$: $\pm 1, \pm 2, \pm 5, \pm 10, \pm 13$

CRAP!

Need a '0' in the x^4 slot!!!

Try $x = 1$:

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & -10 & 70 & -51 & -130 \\ & & & 1 & -9 & 61 & 10 \\ \hline & & 1 & -9 & 61 & 10 & \text{Newp} \end{array}$$

$x = -1$:

$$\begin{array}{r|rrrrr} -1 & 1 & -10 & 70 & -51 & -130 \\ & & -1 & 11 & 81 & -30 \\ \hline & 1 & -11 & 81 & 30 & \text{Newp} \end{array}$$

$x = -2$ (Better work!) DOH!

$$\begin{array}{r|rrrrr} -2 & 1 & -10 & 70 & -51 & -130 \\ & & -2 & 24 & -108 & \\ \hline & 1 & -12 & 94 & \text{Nope} & \end{array}$$

$x = +2$

$$\begin{array}{r|rrrrr} 2 & 1 & -10 & 70 & -51 & -130 \\ & & 2 & -16 & 108 & \\ \hline & 1 & -8 & 54 & 57 & \end{array}$$

$x^5 - 10x^3 + 70x^2 - 51x - 130$

Try $x=2$:

Try 2, again!

$$
\begin{array}{r|rrrrrr}
2 & 1 & 0 & -10 & 70 & -51 & -130 \\
 & & 2 & 4 & -12 & 116 & 130 \\
\hline
2 & 1 & 2 & -6 & 58 & 65 & 0 \\
 & & 2 & 8 & 4 & 128 & \\
\hline
 & 1 & 4 & 2 & 64 & \text{Nope} &
\end{array}
$$

Sweet! Nice!

So, $f(x) = (x-2)(\underline{x^4+2x^3-6x^2+58x+65})$ is where we're at.

Try $x=-1$: (Use Depressed Polynomial)

Problem is now 4^{th} Degree

$$
\begin{array}{r|rrrrr}
-1 & 1 & 2 & -6 & 58 & 65 \\
 & & -1 & -1 & 7 & -65 \\
\hline
-5 & 1 & 1 & -7 & 65 & 0 \\
 & & -5 & 20 & -65 & \\
\hline
 & 1 & -4 & 13 & 0 &
\end{array}
$$

Try $x=-5$:

Sweet! Sweet!

So now, we're here:

$$f(x) = (x-2)(x+1)(x+5)(x^2-4x+13)$$

$x^2-4x+13$ is new depressed polynomial

BREAK IT DOWN!

$a=1,\ b=-4,\ c=13$

$b^2-4ac = (-4)^2-4(1)(13) = 16-52 = -36$

& $x = \dfrac{-(-4) \pm \sqrt{-36}}{2(1)} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$, so

$$f(x) = (x-2)(x+1)(x+5)(x-(2+3i))(x-(2-3i))$$

& f has been split into linear (1^{st}-degree) factors!

TIPS: Any time your guess gives zero remainder, use that guess, again, in case it works twice.

$x^4 + 2x^3 - 15x^2 + 4x + 20$

Rational Zeros:

$x = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

from $\frac{p}{q} = \frac{\text{factor of } 20}{\text{factor of } 1}$

Try $x = +1$ (won't work)

Try $x = -1$

$$\begin{array}{r|rrrrr} -1 & 1 & 2 & -15 & 4 & 20 \\ & & -1 & -1 & 16 & -20 \\ \hline & 1 & 1 & -16 & 20 & 0 \end{array}$$

Sweet!

$(x+1)(x^3+x^2-16x+20)$

Try $x=-1$ again:

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -16 & 20 \\ & & -1 & 0 & 16 \\ \hline & 1 & 0 & -16 & \end{array}$$

Nope

Try $x=2$: (on depressed poly)

$x^3 + x^2 - 16x + 20$

Each zero remainder makes the job 1 degree smaller.

$(x+1)(x-2)(x^2+3x-10)$

$(x+1)(x-2)(x-2)(x+5)$

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -16 & 20 \\ & & 2 & 6 & -20 \\ \hline & 1 & 3 & -10 & 0 \end{array}$$

Sweet!

Try 2, again:

$$\begin{array}{r|rrr} 2 & 1 & 3 & -10 \\ & & 2 & 10 \\ \hline & 1 & 5 & 0 \end{array}$$

Super!

See? $x+5$?

Quadratic Formula would've worked at this point, too

This work says

$x^4 + 2x^3 - 15x^2 + 4x + 20 = (x-1)(x+2)^2(x+5)$

When degree gets down to 2, Quadratic Formula!