We begin with a Take-Home Test and Proceed to an in-class test.

1. (5 pts) For each of the following polynomials, give an end behavior graphic, for instance, $\swarrow \cdots \searrow$

   a. $f(x) = -3x^3 + 7x^2$

   $-3x^3$

   End Behavior

   $x^2, x^4, \ldots$ 2 +

   $-x^2, -x^4, \ldots$

   $x, x^3, x^5, \ldots$

   $-x, -x^3, -x^5, \ldots$

   b. $g(x) = 25x^4 - 15x^2 + 5$

Let $f(x) = 4x^5 - 12x^4 - 5x^3 + 21x^2 - 11x - 21$ for the remainder of this test.

1 2 3

2. (5 pts) What does Descartes' Rule of Signs tell you about positive and negative zeros (roots) of f?

$\boxed{\text{3 or 1 positive zeros.}}$

$f(-x) = -4x^5 - 12x^4 + 5x^3 + 12x^2 + 11x - 21$

1 2

$\boxed{\text{2 or 0 negative zeros}}$

3. (5 pts) Use the Rational Zeros (Roots) Theorem to list the possible rational zeros of f.

$(2x+1)(5x-3) = 10x^2 - x - 3$

$x = -\frac{1}{2}$ $x = \frac{3}{5}$

$\frac{p}{q} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{7}{1}, \pm\frac{7}{2}, \pm\frac{7}{4}, \pm\frac{21}{1}, \pm\frac{21}{2}$
$\pm\frac{21}{4}$

4. (5 pts) Show that $x = 5$ is an upper bound on real zeros for f.

Synthetic Division by $x-5$
Bottom Row all positive → upper bound
or zero

Bottom Row alternating signs (or zero)
→ Lower Bound

$$f(x) = 4x^5 - 12x^4 - 5x^3 + 21x^2 - 11x - 21$$

5	4	-12	-5	21	-11	-21
		20	40	175	Big	Bigger
	4	8	35	196	Big	Bigger

All positive →
$x = 5$ is U.B. on real zeros.

-5	4	-12	-5	21	-11	-21
		-20	160	BIG	-Big	+Big
	4	-32	155	BIG	-Big	Big

Signs alternate →

$x = -5$ is L.B. ..

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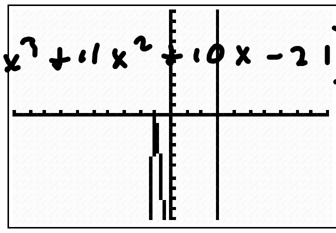
imaginary is what author says. Sucks.

5. (5 pts) Find all real and nonreal zeros of $f(x) = 4x^5 - 12x^4 - 5x^3 + 21x^2 - 11x - 21$. Show the breakdown by synthetic divisions, step by step. Do your work on separate paper, and only show *me* the guesses that *worked*. Neatness counts. No credit for sloppy work.

~~See next page for some of the guesswork. Since THIS part of the test was take-home, that means we can cheat with technology, and I'll show you some of that.~~

Meh. It's a take-home. Slew it with grapher.

Looks like $x = -1$, $x = 3$. $(x+1)(4x^4 - 16x^3 + 11x^2 + 10x - 21)$



$$
\begin{array}{r|rrrrrr}
-1 & 4 & -12 & -5 & 21 & -11 & -21 \\
 & & -4 & 16 & -11 & -10 & +21 \\
\hline
-1 & 4 & -16 & 11 & 10 & -21 & 0 \\
 & & -4 & 20 & -31 & 21 & \\
\hline
3 & 4 & -20 & 31 & -21 & 0 & \\
 & & 12 & -24 & 21 & & \\
\hline
 & 4 & -8 & 7 & 0 & &
\end{array}
$$

Yes!

Yes!

Sweet!

$4x^2 - 8x + 7$ is depressed polynomial

$(x+1)^2(4x^3 - 20x^2 + 31x - 21)$

$$4x^2 - 8x + 7 = 0$$

$$4x^2 - 8x = -7$$

$$4(x^2 - 2x + 1^2) = -7 + 4$$

$$4(x-1)^2 = -3$$

$$(x-1)^2 = -\frac{3}{4}$$

$$x - 1 = \pm\sqrt{-\frac{3}{4}} = \pm i\sqrt{\frac{3}{4}} = \pm i\frac{\sqrt{3}}{\sqrt{4}} = \pm i\frac{\sqrt{3}}{2}$$

$$\boxed{x = 1 \pm \frac{\sqrt{3}}{2}i} = \frac{2 \pm \sqrt{3}\,i}{2}$$

$x = -1$, $m = 2$; $x = 3$, $m = 1$; $x = 1 \pm \frac{\sqrt{3}}{2}i$

$x=-1$, $m=2$; $x=3$, $m=1$; $x = 1 \pm \frac{\sqrt{3}}{2} i$

$\boxed{\text{Leading coefficient is } 4}$

Factor $f(x)$ over the reals (in the real number system)

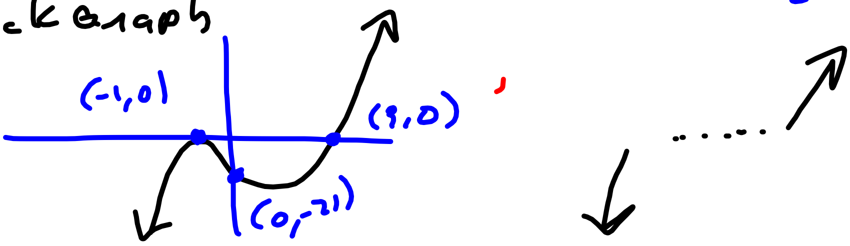
$$f(x) = (x+1)^2(x-3)(4x^2-8x+7)$$

... over the complex numbers:

$$f(x) = 4(x+1)^2(x-3)\left(x-\left(1+\frac{\sqrt{3}}{2}i\right)\right)\left(x-\left(1-\frac{\sqrt{3}}{2}i\right)\right)$$

Quick graph

not asked for.



1. (5 pts) Use synthetic division to find $P(2)$ if $P(x) = 5x^4 + 2x^3 - 3x + 121$.

Think Remainder Theorem!

$$\begin{array}{r|rrrrr} 2 & 5 & 2 & 0 & -3 & 121 \\ & & 10 & 24 & 48 & 90 \\ \hline & 5 & 12 & 24 & 45 & 211 \end{array} \Rightarrow f(2) = 211$$

b/c our work says

$$P(x) = (x-2)(5x^3+2x^2+24x+45) + 211$$

$$\frac{28}{9} = 3\frac{1}{9} = 3 + \frac{1}{9}$$

$$28 = 3(9) + 1$$

$$\Longrightarrow P(2) = 211$$

We do the same thing with $\frac{P(x)}{x-2}$!

3. (5 pts) Multiply (Expand) and simplify the product: $(x-(3-7i))(x-(3+7i))$.

$$(x-(3-7i))(x-(3+7i))$$

$$= x^2 + x(-(3+7i)) + (-(3-7i))(x) + (-(3-7i))(-(3-7i))$$

$$= x^2 + x(-3-7i) + (-3+7i)x + \underline{(3-7i)(3+7i)}$$

$$= x^2 - 3x - 7ix - 3x + 7ix + 9 + 49$$

$$= \boxed{x^2 - 6x + 58}$$

$$(x-3+7i)(x-3-7i)$$

$$= x^2 - 3x - 7ix - 3x + 9 + 21i + 7ix - 21i - 49i^2$$

+49

$$= x^2 - 6x + 58$$

$$(a-b)(a+b) = a^2 - b^2$$

$$(a-bi)(a+bi)$$

$$= a^2 - (bi)^2$$

$$= a^2 - b^2 i^2$$

$$= a^2 + b^2$$

$$(31-10i)(31+10i)$$

$$31^2 + 10^2$$

2. (5 pts) Construct a polynomial (in factored form) of minimal degree that has *real* coefficients (if you expand it, but *don't* expand it!) and the following zeros, with the indicated multiplicities. Do *not* expand..

$x = 1, m = 2;\ x = 3,\ m = 5;\ x = -7, m = 13;\ x = 2 - 3i, m = 1.$

Factor Theorem and Conjugate Pairs Theorem

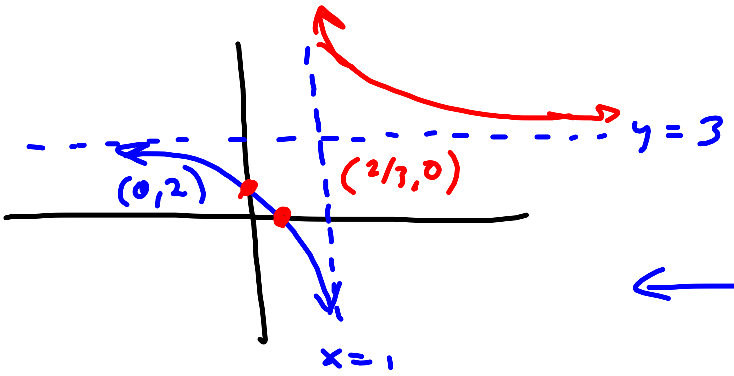
$$(x-1)^2(x-3)^5(x+7)^{13}(x-(2-3i))\underline{(x-(2+3i))}$$

Conjugate Pairs Theorem

4. (5 pts) Sketch the graph of $\frac{3x-2}{x-1}$. Show all asymptotes and intercepts.

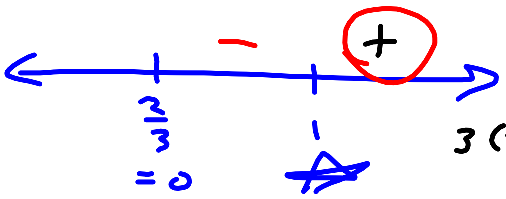
H.A.: $y = 3$ $\qquad \frac{3x}{x} = 3$

V.A.: $x = 1$ makes Denom. zero



$y = 0$: $3x - 2 = 0$

$x = \frac{2}{3}$

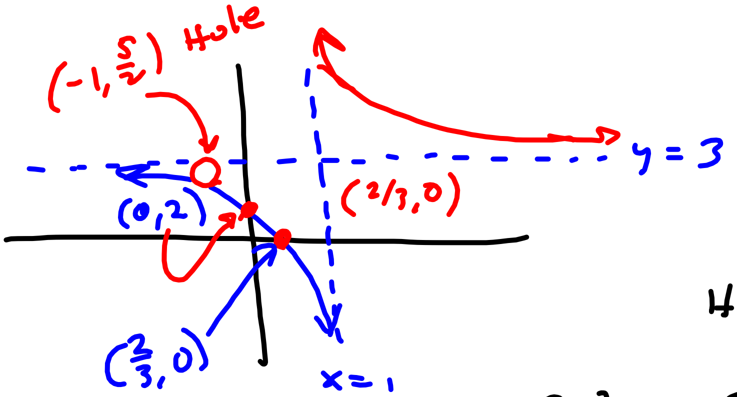


$3(x - \frac{2}{3})'$

$x = 2$ $\qquad \frac{3x-2}{(x-1)}'$

$\frac{3(2)-2}{2-1} = \frac{4}{1} = 4$

5. (5 pts) Based on your work on the previous problem, give a sketch of $\left(\frac{3x-2}{x-1}\right)\cdot\left(\frac{x+1}{x+1}\right)$



$x=-1$ has a hole.

$$\frac{3(-1)-2}{-1-1} = \frac{-3-2}{-2} = \frac{-5}{-2} = \frac{5}{2}$$

Hole @ $(-1, \frac{5}{2})$

$$\frac{3x^2+x-2}{x^2-1} =$$

6. (5 pts) Solve the inequality $x^2 - 7x - 11 > 0$. Give answer in set-builder and interval notation.

$$x^2 - 7x + \left(\frac{7}{2}\right)^2 = 11 + \frac{49}{4} = \frac{44}{4} + \frac{49}{4} = \frac{93}{4}$$

$$\left(x - \frac{7}{2}\right)^2 = \frac{93}{4}$$

$$x - \frac{7}{2} = \pm\sqrt{\frac{93}{4}} = \pm\frac{\sqrt{93}}{2}$$

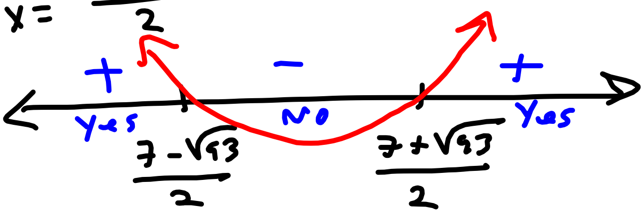
$$\sqrt{\left(x-\frac{7}{2}\right)^2} = \sqrt{\frac{93}{4}}$$

$$\left|x - \frac{7}{2}\right| = \sqrt{\frac{93}{4}}$$

$$x = \frac{7 \pm \sqrt{93}}{2}$$

$x^2 - 7x - 11$

Sign Pattern



$$\left(-\infty, \frac{7-\sqrt{93}}{2}\right) \cup \left(\frac{7+\sqrt{93}}{2}, \infty\right)$$

7. (10 pts) Let $f(x)=\dfrac{x-3}{x-5}$ and $g(x)=\sqrt{x-7}$. Form the composite function $(f\circ g)(x)$. Do not simplify.

What is the domain of $f\circ g$?

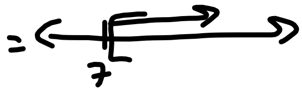
$\mathcal{D}(f)$: Need $x-5\neq 0$

$\{x \mid x\neq 5\}$

$= (-\infty,5)\cup(5,\infty)$

$\mathcal{D}(g)$: $x-7\geq 0$

$\{x \mid x\geq 7\}$

$=$ 

$= [7,\infty)$

$(f\circ g)(x) = f(g(x)) =$

$$= \frac{g(x)-3}{g(x)-5} = \frac{\sqrt{x-7}-3}{\sqrt{x-7}-5}$$

$\mathcal{D}(f\circ g) = \{x \mid x\in\mathcal{D}(g) \text{ and } g(x)\in\mathcal{D}(f)\}$

$= \{x \mid x\geq 7 \text{ and } g(x)\neq 5\}$

$= \{x \mid x\geq 7 \text{ and } x\neq 32\}$

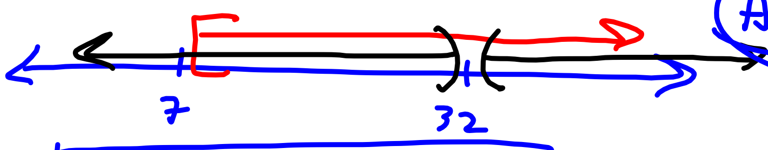
$g(x)\neq 5$

$\sqrt{x-7}\neq 5$

$(\sqrt{x-7})^2\neq 5^2$

$x-7\neq 25$

$x\neq 32$

$=$ 

AND

$= \boxed{[7,32)\cup(32,\infty)}$

8. Let $f(x) = 4x^5 \underset{1}{-} 16x^4 \underset{2}{+} 49x^3 + 11x^2 \underset{3}{-} 35x - 13$

a. (10 pts) Find all possible rational zeros.

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 13, \pm \frac{13}{2}, \pm \frac{13}{4}$

b. (10 pts) Use Descartes' Rule to find the possible number of positive and negative zeros of f.

3 or 1 positives

$f(-x) = -4x^5 - 16x^4 - 49x^3 \underset{1}{+} 11x^2 + 35x \underset{2}{-} 13$

2 or 0 negative zeros

c. (10 pts) Use synthetic division to show that $x = 4$ is an upper bound on real zeros for f.

8. Let $f(x)=4x^5-16x^4+49x^3+11x^2-35x-13$

d. (10 pts) Find all real zeros of f. Then factor f over the field of real numbers.

Depressed Poly

$$\begin{array}{r|rrrrrr} 1 & 4 & -16 & 49 & 11 & -35 & -13 \\ & & 4 & -12 & 37 & 48 & 13 \\ \hline -1 & 4 & -12 & 37 & 48 & 13 & 0 \\ & & -4 & 16 & -53 & & \\ \hline & 4 & -16 & 53 & & & \end{array}$$

Nice

$(x-1)\underbrace{(4x^4-12x^3+37x^2+48x+13)}_{\text{Yugh!}}$

But went too far!

This one was Impossible to get very far.

This is as far as this one goes.

$\boxed{\text{x=1 is only real zero I found}}$

8. Let $f(x) = 9x^5 - 21x^4 + 10x^3 + 14x^2 - 19x + 7$

e. (10 pts) Find the remaining nonreal zeros of f and factor f over the field of complex numbers.

Better-posed problem

$\pm 1, \pm \frac{1}{3}, \pm \frac{1}{9}, \pm 7, \pm \frac{7}{3}, \pm \frac{7}{9}$

1	9	-21	10	14	-19	7	
		9	-12	-2	12	-7	
1	9	-12	-2	12	-7	0	Good!
		9	-3	-5	7		
-1	9	-3	-5	7	0		Good!
		-9	12	-7			
	9	-12	7	0			Sweet!

$9x^2 - 12x + 7$ is new depressed polynomial

$a = 9, b = -12, c = 7$

$b^2 - 4ac = (-12)^2 - 4(9)(7)$

$= 144 - 252$

$= -108$

$2 | 108$, $2 | 54$, $3 | 27$, $3 | 9$, 3

$\sqrt{108} = 2 \cdot 3\sqrt{3} = 6\sqrt{3}$

No real sol'n, so answer to previous question is $(x-1)^2(x+1)(9x^2 - 12x + 7)$ is factored over the real numbers

To answer This question, we find zeros of $9x^2 - 12x + 7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{12 \pm \sqrt{-108}}{2(9)} = \frac{12 \pm 6i\sqrt{3}}{18}$$

$$= \frac{6(2 \pm i\sqrt{3})}{18} = \boxed{\frac{2 \pm i\sqrt{3}}{3} = x}$$ nonreal zeros.

$P(x) =$

$$\boxed{9(x-1)^2(x+1)\left(x - \left(\frac{2+i\sqrt{3}}{3}\right)\right)\left(x - \left(\frac{2-i\sqrt{3}}{3}\right)\right)}$$

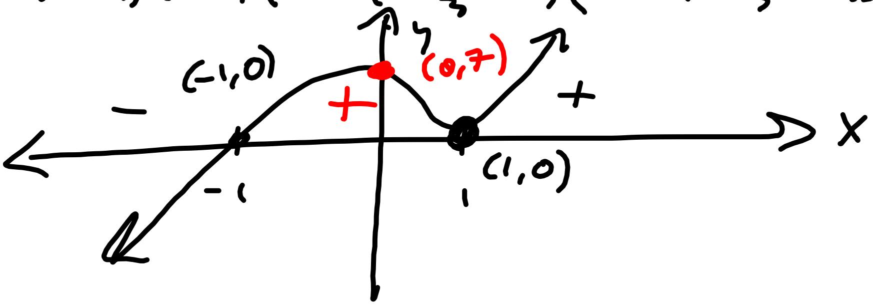
$= (x-1)^2(x+1)(3x - (2+i\sqrt{3}))(3x - (2-i\sqrt{3}))$

9. (10 pts) Based on your work in the previous problem, provide a rough sketch of f.

$9x^5$

$P(x) =$

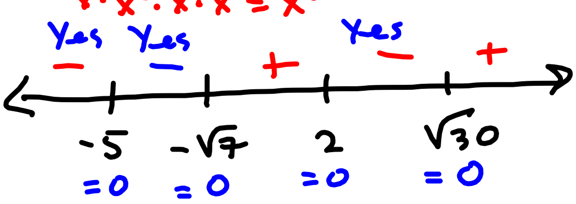
$$9(x-1)^2(x+1)\left(x-\left(\frac{2+i\sqrt{3}}{3}\right)\right)\left(x-\left(\frac{2-i\sqrt{3}}{3}\right)\right)$$



a. Solve the inequality:

$$(x-2)(x+5)^2\left(x-\sqrt{30}\right)\left(x+\sqrt{7}\right)\le 0$$

$x\cdot x^2\cdot x\cdot x = x^5$



End behavior

Analyze sign changes.

$(-\infty,-5]\cup[-5,-\sqrt{7}]\cup[2,\sqrt{30}]$

$\boxed{(-\infty,-\sqrt{7}]\cup[2,\sqrt{30}]}$

$$\frac{\left(x-\sqrt{30}\right)\left(x+\sqrt{7}\right)}{(x-2)(x+5)^2}\le 0$$

Same sign pattern, but throw out $x=-5, +2$

$\boxed{(-\infty,-5)\cup(-5,-\sqrt{7}]\cup(2,\sqrt{30}]}$