The Fundamental Theorem of Algebra

> If $y = P(x)$ is a polynomial function of positive degree, then $y = P(x)$ has at least one zero in the set of complex numbers.

And if we use the factor theorem to "split off the linear factor (x - c) corresponding to one of the zeros," we get a *depressed polynomial*, of one degree less than $P(x)$'s degree. We then apply FTA to the depressed polynomial, use the factor theorem (and synthetic division) to split off another linear factor, until we've split off *every* linear factor, and a splitting of $P(x)$ into linear factors is achieved.

In class, last time, (and time before) I told you that FTA says that a polynomial of degree n has n complex zeros. And that's what always having at least *one* means, once you have the Factor Theorem under your belt!

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$= a_n (x - c_1) \cdot (x - c_2) \cdot (x - c_3) \cdots (x - c_n)$$

$P(x) = 3x^2 - x - 2$ $\qquad$ $(3x+2)(x-1)$

$n = 2$ $\qquad$ $= 3x^2 - x - 2 = 0$

$P(x) = 3(x - (-\frac{2}{3}))(x-1)$ $\Longrightarrow x = -\frac{2}{3}, 1$

$3(x - c_1)(x - c_2)$

$$P(x) = 10x^3 + 21x^2 + 5x - 6$$

$$P(x) = 10(x+1)(x - \tfrac{2}{5})(x + \tfrac{3}{2})$$

Example build for Rational Zeros Theorem and Descartes' Rule of Signs. These "educate" our guesses at what the zeros *might* be. We check our guesses with synthetic division and when we guess right, we've found a zero *and* we've lowered the degree of the polynomial we're working with by one degree!

$2 \cdot x + 3 = P(x)$

$a_1 x + a_0$

$2x + 3$

$P(x) = 0 \Rightarrow$

$x = -\frac{3}{2}$

Note:

3 divides $a_0 = 3$

2 divides $a_1 = 2$

$\% \cdot (x+1)$

$(2x+3)(x+1)$

$x = -3/2, -1$

expand(%)

$2x^2 + 5x + 3$

$\% \cdot (5 \cdot x - 2)$

$(2x^2 + 5x + 3)(5x - 2)$

$x = -3/2, -1, 2/5$

expand(%)

$10x^3 + 21x^2 + 5x - 6$

$\frac{p}{q} = \frac{2}{5}$ → a factor of -6 / → a factor of 10

So what? We want to find ALL the zeros of, say, $P(x) = 10x^3 + 21x^2 + 5x - 6$

$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{5}, \pm \frac{1}{10},$$
$$\pm 2, \cancel{\pm \frac{2}{2}}, \pm \frac{2}{5}, \cancel{\pm \frac{2}{10}}$$
$$\pm 3, \pm \frac{3}{2}, \pm \frac{3}{5}, \pm \frac{3}{10}$$
$$\pm 6, \cancel{\pm \frac{6}{2}}, \pm \frac{6}{5}, \cancel{\pm \frac{6}{10} = \pm \frac{3}{5}}$$

Guess $x=1$:

$$\begin{array}{r|rrrr} 1 & 10 & 21 & 5 & -6 \\ & & 10 & 31 & 36 \\ \hline & 10 & 31 & 36 & 30 \end{array}$$

Bottom row Nonnegative

Bounds on Real Zeros tells you you're DONE looking past $x=1$ (Nothing to the right of $x=1$ is a zero.

$$\begin{array}{r|rrrr} -1 & 10 & 21 & 5 & -6 \\ & & -10 & -11 & 6 \\ \hline & 10 & 11 & -6 & 0 \\ & x^2 & x' & c & r \end{array}$$

sweet!

This says $x=-1$ is a root.

" " $x+1$ is a factor.

" " I can split off a factor of $x+1$

Depressed polynomial

$P(x) = (x+1)(10x^2+11x-6)$

Depressed eq'n

$10x^2+11x-6=0$

$\sqrt{}$

$a=10, b=11, c=-6$

$b^2-4ac = 11^2-4(10)(-6)$

$121+240$

$=361 \rightarrow \sqrt{361} = 19$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-11 \pm 19}{2(10)} = \dfrac{-11 \pm 19}{20}$

$\frac{2}{5}$ $\quad$ $-\frac{3}{2}$

So, zeros of $P(x)$ are $x=-1, \frac{2}{5}, -\frac{3}{2}$ and

$P(x) = 10(x+1)(x-\frac{2}{5})(x+\frac{3}{2})$

$P(x) = 10x^3+21x^2+5x-6$

Theorem on Bounds

Suppose that $P(x)$ is a polynomial with real coefficients and a positive leading coefficient, and synthetic division with $x - c$ is performed.

- If $c > 0$ and all terms in the bottom row are nonnegative, then c is an upper bound for the roots of $P(x) = 0$.
- If $c < 0$ and the terms in the bottom row alternate in sign, then c is a lower bound for the roots of $P(x) = 0$.

3.3 #65 is a good example for this theorem.

$P(x) = 10x^3 + 21x^2 + 5x - 6$

we saw an upper bound example, already. Now for a lower bound:

$x = -1, -\frac{3}{2}, \frac{2}{5}$

Try $x = -3$

$$\begin{array}{r|rrrr} -3 & 10 & 21 & 5 & -6 \\ & & -30 & 27 & -96 \\ \hline & 10 & -9 & 32 & -102 \end{array}$$

signs alternate, so we know we don't need to check $x = -6$,

Conjugate Pairs Theorem

If $P(x) = 0$ is a polynomial equation with real coefficients and the complex number $a + bi$ $(b \neq 0)$ is a root, then $a - bi$ is also a root.

I'm not so sure this is a really practical tool for breaking down a polynomial, in practice. But it's GREAT for building a polynomial in factored form that turns out to have real coefficients when you expand it. I use this theorem all the time, when I'm building examples for class. Also, it's really easy to write test questions for this concept!

$(x-2+3\cdot I)(x-2-3\cdot I)\cdot(3\cdot x-2)\cdot(x-1)$

$(x-2+3\,\mathrm{I})\,(x-2-3\,\mathrm{I})\,(3\,x-2)\,(x-1)$

expand(%)

$3x^4-17x^3+61x^2-73x+26$

$(x-2+3i)(x-2-3i)$
I built it to have $x=-2\pm3i$ as roots.

Rat'l zeros:
$\pm 1, \pm\frac{1}{3}$
$\pm 2, \pm\frac{2}{3}$
$\pm 13, \pm\frac{13}{3}$
$\pm 26, \pm\frac{26}{3}$

Write a polynomial (in factored form) of degree 2, with REAL coefficients and zeros (roots) @ $x=2+i$, 3

$(x-3)(x-(2+i))(x-(2-i))$

IMPOSSIBLE! By conjugate pairs theorem.

Descartes's Rule of Signs

Suppose $P(x) = 0$ is a polynomial equation with real coefficients and with terms written in descending order.

- The number of positive real roots of the equation is either equal to the number of variations of sign of $P(x)$ or less than that by an even number.
- The number of negative real roots of the equation is either equal to the number of variations of sign of $P(-x)$ or less than that by an even number.

$f := x \rightarrow 2 \cdot x^3 - 5 \cdot x^2 - 6 \cdot x + 4$

$x \rightarrow 2x^3 - 5x^2 - 6x + 4$

$solve(f(x) = 0, x)$

$\frac{1}{2}, \sqrt{5} + 1, -\sqrt{5} + 1$

$factor(f(x))$

$(2x - 1)\underline{(x^2 - 2x - 4)}$

Irreducible quadratic factor over the RATIONALS.

$\pm 1, \pm \frac{1}{2}, \pm 2, \pm 4$

$x = 1$ Nah

$x = -1$ Nah

$x = 2$:

$$\begin{array}{r|rrrr} 2 & 2 & -5 & -6 & 4 \\ & & 4 & -2 & \text{Nah} \\ \hline & 2 & -1 & -8 & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 2 & -5 & -6 & 4 \\ & & -4 & 18 & -24 \\ \hline & 2 & -9 & 12 & -20 \end{array}$$

Ditch $x = -4$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & -6 & 4 \\ & & 1 & -2 & -4 \\ \hline & 2 & -4 & -8 & 0 \end{array}$$ sweet!

$f(x) = (x - \frac{1}{2})(2x^2 - 4x - 8)$

$= 2(x - \frac{1}{2})(x^2 - 2x - 4)$ Depressed polynomial

$x^2 - 2x - 4 = 0 \leftarrow$ Now solve depressed eq'n

Oct 9-9:15 AM

$x^2 - 2x + 1^2 = 4 + 1$

$(x-1)^2 = 5$

$x - 1 = \pm\sqrt{5}$

$x = 1 \pm \sqrt{5}$

$f(x) = 2\left(x - \tfrac{1}{2}\right)\left(x - (1+\sqrt{5})\right)\left(x - (1-\sqrt{5})\right)$

Real, but irrational.

This suggests kind of a "conjugate pairs theorem" for polynomials with RATIONAL coefficients

$f(x) = 2x^3 - 5x^2 - 6x + 4$

Rational coefficients,

so once you get $x = 1+\sqrt{5}$ is a zero you know $x = 1-\sqrt{5}$ is, too.

Descartes's Rule of Signs

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- The number of negative real roots of the equation is either equal to the number of variations of sign of $P(-x)$ or less than that by an even number.

$f := x \rightarrow 2 \cdot x^3 - 5 \cdot x^2 - 6 \cdot x + 4$

$x \rightarrow 2x^3 - 5x^2 - 6x + 4$

$solve(f(x) = 0, x)$

$\frac{1}{2}, \sqrt{5} + 1, -\sqrt{5} + 1$

$factor(f(x))$

$(2x - 1)(x^2 - 2x - 4)$

$f(x) = 2x^3 - 5x^2 - 6x + 4$
1 2

There are 2 or 0 positive zeros

$f(-x) = 2(-x)^3 - 5(-x)^2 - 6(-x) + 4$
$= -2x^3 - 5x^2 + 6x + 4$

1 sign change.
Exactly 1 negative zero.

3.3 #5 15, 16, 18, 23, 26, 28, 62, 67

Midterm
C1, C2, plus
one from C3
FRIDAY. → Monday

On tests, you will NOT expand these types of questions

$x^4+2x^3-3x^2-4x+4=0$ Solve

$f(x)=x^4+2x^3-3x^2-4x+4$ Find zeros

$f(x)=x^4+\underbrace{+2x^3-3x^2}_{1}\underbrace{-4x+4}_{2}$ Split into the product of linear factors.

Rat'l Zeros: $\pm1, \pm2, \pm4 = \frac{p}{q}$

$p: \pm1, \pm2, \pm4$

$q: \pm1$

$-3(-x)^2=-3x^2$

Descartes: 2 or 0 positive zeros.

$f(-x)=\underbrace{x^4-2x^3}\underbrace{-3x^2+4x}+4$ 2 or 0 negative zeros

$\cancel{(x-1)}(\cancel{x^3+3x^2-4})=f(x)$

$x-1=0$

$x=1$

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & -3 & -4 & 4 \\ & & 1 & 3 & 0 & -4 \\ \hline 1 & 1 & 3 & 0 & -4 & 0 \\ & & 1 & 4 & 4 & \\ \hline & 1 & 4 & 4 & 0 & \end{array}$$

Sweet.

sweet

$f(x)=(x-1)(x-1)(x^2+4x+4)$

x^2+4x+4
$=(x+2)^2 \Longrightarrow f(x)=(x-1)^2(x+2)^2$

All nonnegative.

$x=1$ multiplicity $=2$
$x=-2$ " "

$x=1$ is upper bound on real zeros!

This one had Repeated Roots:

Always check for them.

After we got $(x-1)(x^3+3x^2-4)$, we're Done messing with $f(x)=x^4+2x^3-3x^2-4x+4$ and are ONLY concerned with breaking-down x^3+3x^2-4 the rest of the way.

Oct 9-9:39 AM