The Fundamental Theorem of Algebra

If $y = P(x)$ is a polynomial function of positive degree, then $y = P(x)$ has at least one zero in the set of complex numbers.

And if we use the factor theorem to "split off the linear factor (x - c) corresponding to one of the zeros," we get a *depressed polynomial*, of one degree less than $P(x)$'s degree. We then apply FTA to the depressed polynomial, use the factor theorem (and synthetic division) to split off another linear factor, until we've split off *every* linear factor, and a splitting of $P(x)$ into linear factors is achieved.

In class, last time, (and time before) I told you that FTA says that a polynomial of degree n has n complex zeros. And that's what always having at least *one* means, once you have the Factor Theorem under your belt!

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$= a_n(x - c_1)\cdot(x - c_2)\cdot(x - c_3)\cdots(x - c_n)$$

$P(x) = 3x^2 - x - 2$

$n = 2$

$P(x) = 3(x - (-\frac{2}{3}))(x - 1)$

$3(x - c_1)(x - c_2)$

$(3x+2)(x-1)$

$= 3x^2 - x - 2 = 0$

$\Rightarrow x = -\frac{2}{3}, 1$

$P(x) = 10x^3 + 21x^2 + 5x - 6$

$P(x) = 10(x+1)(x - \frac{2}{5})(x + \frac{3}{2})$

Example build for Rational Zeros Theorem and Descartes' Rule of Signs. These "educate" our guesses at what the zeros *might* be. We check our guesses with synthetic division and when we guess right, we've found a zero *and* we've lowered the degree of the polynomial we're working with by one degree!

$2\cdot x+3 = P(x)$

$a_1 x + a_0$

$2x+3$

$P(x)=0 \Rightarrow x=-\frac{3}{2}$

Note:

3 divides $a_0=3$

2 divides $a_1=2$

$\%\cdot(x+1)$

$(2x+3)(x+1)$

$x=-3/2, -1$

$\text{expand}(\%)$

$2x^2+5x+3$

$\%\cdot(5\cdot x-2)$

$(2x^2+5x+3)(5x-2)$

$x=-3/2, -1, 2/5$

$\text{expand}(\%)$

$10x^3+21x^2+5x-6$

$\frac{p}{q}=\frac{2}{5}$ → a factor of -6 / → a factor of 10

So what? We want to find ALL the zeros of, say, $P(x)=10x^3+21x^2+5x-6$

$$\frac{p}{q}=\pm 1, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10},$$
$$\pm 2, \pm\cancel{\frac{2}{2}}, \pm\frac{2}{5}, \pm\cancel{\frac{2}{10}}$$
$$\pm 3, \pm\frac{3}{2}, \pm\frac{3}{5}, \pm\frac{3}{10}$$
$$\pm 6, \pm\cancel{\frac{6}{2}}, \pm\frac{6}{5}, \pm\cancel{\frac{6}{10}=\pm\frac{3}{5}}$$

Guess $x=1$:

$$\begin{array}{r|rrrr} 1 & 10 & 21 & 5 & -6 \\ & & 10 & 31 & 36 \\ \hline & 10 & 31 & 36 & 30 \end{array}$$

Bottom row Nonnegative

Bounds on Real Zeros tells you you're DONE looking past $x=1$ (Nothing to the right of $x=1$ is a zero.

$$\begin{array}{r|rrrr} -1 & 10 & 21 & 5 & -6 \\ & & -10 & -11 & 6 \\ \hline & 10 & 11 & -6 & 0 \\ & x^2 & x^1 & c & r \end{array}$$

sweet!

This says $x=-1$ is a root.

" " $x+1$ is a factor.

" " I can split off a factor of $x+1$

Depressed polynomial

$P(x) = (x+1)(10x^2+11x-6)$

Depressed eq'n

$10x^2+11x-6=0$

$\checkmark$

$a=10, b=11, c=-6$

$b^2-4ac = 11^2-4(10)(-6)$

$121+240$

$=361 \longrightarrow \sqrt{361} = 19$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-11 \pm 19}{2(10)} = \frac{-11 \pm 19}{20}$$

$\frac{2}{5}$ $\quad -\frac{3}{2}$

So, zeros of $P(x)$ are $x=-1, \frac{2}{5}, -\frac{3}{2}$ and

$P(x) = 10(x+1)(x-\frac{2}{5})(x+\frac{3}{2})$

$P(x) = 10x^3+21x^2+5x-6$

Theorem on Bounds

Suppose that $P(x)$ is a polynomial with real coefficients and a positive leading coefficient, and synthetic division with $x - c$ is performed.

- If $c > 0$ and all terms in the bottom row are nonnegative, then c is an upper bound for the roots of $P(x) = 0$.
- If $c < 0$ and the terms in the bottom row alternate in sign, then c is a lower bound for the roots of $P(x) = 0$.

3.3 #65 is a good example for this theorem.

$P(x) = 10x^3 + 21x^2 + 5x - 6$

we saw an upper bound example, already. Now for a lower bound:

$x = -1, -\frac{3}{2}, \frac{2}{5}$

Try $x = -3$

$$\begin{array}{r|rrrr} -3 & 10 & 21 & 5 & -6 \\ & & -30 & 27 & -96 \\ \hline & 10 & -9 & 32 & -102 \end{array}$$

signs alternate, so we know we don't need to check $x = -6$, Conjugate Pairs Theorem

> If $P(x) = 0$ is a polynomial equation with real coefficients and the complex number $a + bi$ $(b \neq 0)$ is a root, then $a - bi$ is also a root.

I'm not so sure this is a really practical tool for breaking down a polynomial, in practice. But it's GREAT for building a polynomial in factored form that turns out to have real coefficients when you expand it. I use this theorem all the time, when I'm building examples for class. Also, it's really easy to write test questions for this concept!

$(x-2+3\cdot I)\,(x-2-3\cdot I)\cdot(3\cdot x-2)\cdot(x-1)$

$(x-2+3\,\mathrm{I})\;(x-2-3\,\mathrm{I})\;(3\,x-2)\;(x-1)$

$expand(\%)$

$3\,x^4-17\,x^3+61\,x^2-73\,x+26$

$(x-2+3i)(x-2-3i)$

I built it to have $x=-2\pm 3i$ as roots.

Rat'l zeros:

$\pm 1,\ \pm\frac{1}{3}$

$\pm 2,\ \pm\frac{2}{3}$

$\pm 13,\ \pm\frac{13}{3}$

$\pm 26,\ \pm\frac{26}{3}$

Write a polynomial (in factored form) of degree 2, ~~with REAL coefficients~~ and zeros (roots) @ $x = 2+i,\ 3$

$(x-3)(x-(2+i))(x-(2-i))$

IMPOSSIBLE! By conjugate pairs theorem.

Oct 9-9:14 AM

Descartes's Rule of Signs

Suppose $P(x) = 0$ is a polynomial equation with real coefficients and with terms written in descending order.

- The number of positive real roots of the equation is either equal to the number of variations of sign of $P(x)$ or less than that by an even number.
- The number of negative real roots of the equation is either equal to the number of variations of sign of $P(-x)$ or less than that by an even number.

$f := x \to 2\cdot x^3 - 5\cdot x^2 - 6\cdot x + 4$ $\quad x \to 2x^3 - 5x^2 - 6x + 4$

$solve(f(x) = 0, x)$ $\quad \frac{1}{2}, \sqrt{5}+1, -\sqrt{5}+1$

$factor(f(x))$ $\quad (2x-1)(x^2-2x-4)$

Irreducible quadratic factor over the RATIONALS.

$\pm 1, \pm \frac{1}{2}, \pm 2, \pm 4$

$x=1$ Nah

$x=-1$ Nah

$x=2$:

$$\begin{array}{r|rrrr} 2 & 2 & -5 & -6 & 4 \\ & & 4 & -2 & \text{Nah} \\ \hline & 2 & -1 & -8 & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 2 & -5 & -6 & 4 \\ & & -4 & 18 & -24 \\ \hline & 2 & -9 & 12 & -20 \end{array}$$

Ditch $x=-4$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & -6 & 4 \\ & & 1 & -2 & -4 \\ \hline & 2 & -4 & -8 & 0 \end{array}$$

sweet!

$f(x) = (x-\frac{1}{2})(2x^2-4x-8)$ Depressed polynomial

$= 2(x-\frac{1}{2})(x^2-2x-4)$

$x^2-2x-4=0$ ← Now solve depressed eq'n

$x^2-2x+1^2 = 4+1$

$(x-1)^2 = 5$

$x-1 = \pm\sqrt{5}$

$x = 1 \pm \sqrt{5}$

$f(x) = 2(x-\frac{1}{2})(x-(1+\sqrt{5}))(x-(1-\sqrt{5}))$

This suggests kind of a "conjugate pairs theorem" for polynomials with RATIONAL coefficients

Real, but irrational.

$f(x) = 2x^3-5x^2-6x+4$

Rational coefficients, so once you get $x = 1+\sqrt{5}$ is a zero you know $x = 1-\sqrt{5}$ is, too.

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- The number of negative real roots of the equation is either equal to the number of variations of sign of $P(-x)$ or less than that by an even number.

$f := x \to 2 \cdot x^3 - 5 \cdot x^2 - 6 \cdot x + 4$

$x \to 2x^3 - 5x^2 - 6x + 4$

$solve(f(x) = 0, x)$

$\frac{1}{2}, \sqrt{5} + 1, -\sqrt{5} + 1$

$factor(f(x))$

$(2x - 1)(x^2 - 2x - 4)$

$f(x) = 2x^3 - 5x^2 - 6x + 4$

1 2

There are 2 or 0 positive zeros

$f(-x) = 2(-x)^3 - 5(-x)^2 - 6(-x) + 4$

$= -2x^3 - 5x^2 + 6x + 4$

1 sign change.
Exactly 1 negative zero.

3.3 #s 15, 16, 18, $\boxed{23, 26, 28,}$ 62, 67

On tests, you will NOT expand these types of questions

Midterm
Ch 1, Ch 2, plus one from Ch 3
FRIDAY.

Oct 9-9:39 AM

Oct 9-9:18 AM