PRACTICE TEST #3

1. State whether the function is a polynomial or not. If not, give a reason why. (probably won't see this on test, but still good to know!) Need non-negative integer powers of x

a. $f(x) = \sqrt{x^2+5} - 14x^3$ $= (x^2+5)^{1/2} - 14x^3$ → not an integer
Not a polynomial

b. $f(x) = x^3 + 3x^2 + \frac{1}{x}$ $= x^3 + 3x^2 + x^{-1}$ → negative; Not a polynomial

c. $f(x) = \frac{3x^3+9(x-3)^2}{3}$ $= \frac{3x^3}{3} + \frac{9(x^2-6x+9)}{3}$ is a polynomial

2. Form a polynomial with real coefficients that has the given zeros and degree.

remember, factored form is $(x-c)$

a. Zeros: 3, multiplicity 2; -2, multiplicity 3; 6, multiplicity 1. Degree 6

$G(x) = (x-3)^2(x+2)^3(x-6)$

b. Zeros: 2, multiplicity 1; 5, multiplicity 2; 3 + 2i, multiplicity 1. Degree 5

$R(x) = (x-2)(x-5)^2(x-(3+2i))(x-(3-2i))$ → these guys always come in conjugate pairs

3. Expand $(x-(2+5i))(x-(2-5i))$

$i^2 = -1$

$x^2 - x(2-5i) - x(2+5i) + (2+5i)(2-5i)$

$x^2 - 2x + \cancel{5ix} - 2x - \cancel{5ix} + (4 - 10i + 10i - 25i^2)$

$x^2 - 4x + 4 - 25(-1)$

$\boxed{x^2 - 4x + 29}$

4. Let $f(x) = 3(x-2)^3(x+4)(x-5)^2$

a. List each real zero and its multiplicity. Determine whether the graph of $f(x)$ touches or crosses the x-axis at each x-intercept.

$(x-2)^3=0$
$x-2=0$
$x=2$
$m=3$, odd
cross

$(x+4)=0$
$x=-4$
$m=1$, odd
cross

$(x-5)^2=0$
$x-5=0$
$x=5$
$m=2$, even
touch

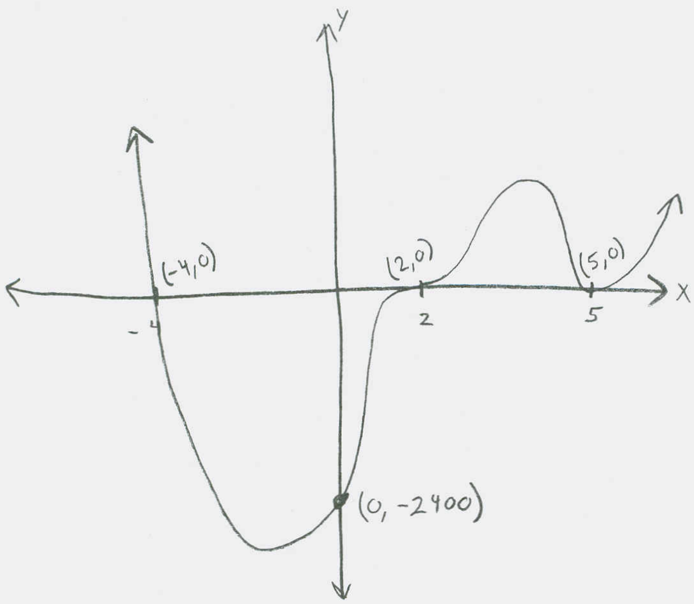
zeros: $x=-4, 2, 5$

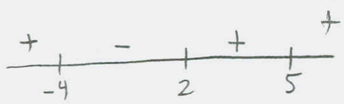
b. Determine the power function that $f(x)$ resembles as $x \to \pm\infty$. This is the End Behavior part of the question. (i.e. determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$)

$3(x)^3(x)(x)^2 = 3x^6 \rightsquigarrow {}^{+}\nwarrow \ldots \nearrow^{+}$

$\lim_{x\to\pm\infty} f(x) = \infty$

d. Use the information you reported to obtain a rough graph of $f(x)$.





y-int:

$f(0) = 3(0-2)^3(0+4)(0-5)^2$

$= 3(-8)(4)(25)$

$= -2400$

y-int: $(0, -2400)$

5. Find the asymptotes (i.e. vertical, and/or horizontal and/or oblique). Reminder: you find the vertical asymptotes by finding where the denominator equals zero. For Part ii, you will need to use long division to find the slant asymptote.

i) $R(x) = \frac{120x^4+5594x^2-0.009x+2}{-12x^4+x^3}$

deg num: 4

deg denom: 4

$$\frac{120x^4}{-12x^4} = -10$$

$\boxed{y=-10 \text{ is horizontal asymptote}}$

$-12x^4+x^3 = x^3(-12x+1)$ set $=0$

$x^3(1-12x)=0$

$1-12x=0 \rightarrow 12x=1$

$x=\frac{1}{12}$

$x^3=0$

$x=0$

$\boxed{\text{V.A.: } x=0,\ x=\frac{1}{12}}$

ii) $G(x) = \frac{x^3-8}{x^2-5x+6} = \frac{(x-2)(x^2+2x+4)}{(x-2)(x-3)} = \frac{x^2+2x+4}{x-3}$ with a hole at $x=2$

O.A.:

$$\begin{array}{r|rrr} 2 & 1 & 2 & 4 \\ & & 2 & 8 \\ \hline & 1 & 4 & 12 \end{array}$$

$x+4+\frac{12}{x-2}$

$\boxed{\text{O.A.: } y=x+4}$

V.A.: find where denominator equals zero.

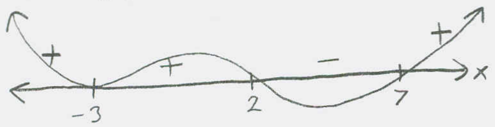
$x-3=0$

$\boxed{x=3}$

6. Solve the inequalities (Hint: use a sign patter.)

a. $(x-2)(x+3)^2(x-7)^3 \geq 0$

Zeros: $x=2$, $m=1$, cross; $x=-3$, $m=2$, touch; $x=7$, $m=3$, cross



E.B. $x \cdot x^2 \cdot x^3 = x^6$ ↖ . . . ↗

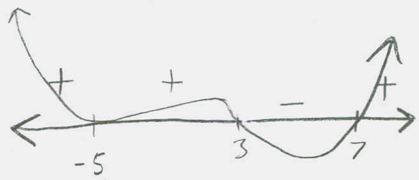
$$\boxed{x \in (-\infty, 2] \cup [7, \infty)}$$

b. $\dfrac{x-3}{(x+5)^2(x-7)^3} \geq 0$

zeros: $x=3$, $m=1$, cross

V.A.: $x=-5$, $m=2$, No sign change

$x=7$, $m=3$, Change sign



$x \in [-\infty, 3) \cup [7, \infty)$

Check: $x=0$

$\dfrac{-3}{(5)^2(-7)^3} = \dfrac{-3}{-(5)^2(7)^3}$ positive

7. Graph the function $R(x) = \dfrac{x^3-3x^2-13x+15}{x^3-5x^2-14x+16} = \dfrac{\cancel{(x-1)}(x+3)(x-5)}{(x+2)\cancel{(x-1)}(3x-8)}$. Key features are asymptotes, holes (if any) and intercepts.

hole at $x=1$

Domain $\mathcal{D} = \{x \mid x \neq -2 \cap x \neq 1 \cap x \neq \frac{8}{3}\}$

Factored Form $R(x) = \dfrac{(x+3)(x-5)}{(x+2)(3x-8)}$

set numerator $= 0$

Zeros: $x=-3$, $m=1$; $x=5$, $m=1$

Y-int: $R(0) = \dfrac{(0+3)(0-5)}{(0+2)(3\cdot 0-8)}$

$= \dfrac{-15}{2(-8)}$

$= \dfrac{-15}{-16} = \dfrac{15}{16}$

Y-int: $\left(0, \frac{15}{16}\right)$

Hole @ $x=1$

$R(1) = \dfrac{(1+3)(1-5)}{(1+2)(3\cdot 1-8)}$

$= \dfrac{4(-4)}{(3)(-5)}$

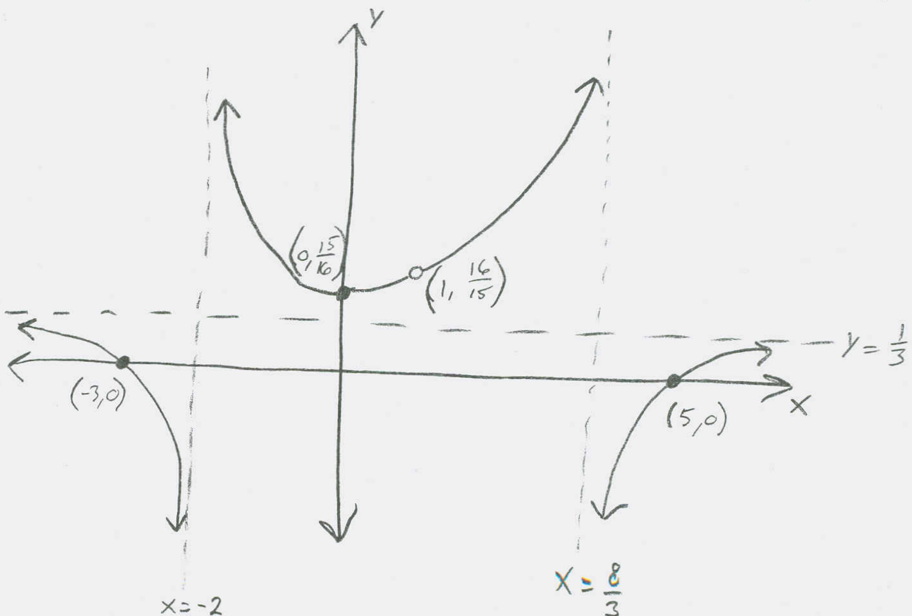
$R(1) = \dfrac{-16}{-15} = \dfrac{16}{15}$

hole: $\left(1, \frac{16}{15}\right)$

V.A.: $x=-2$, $x=\frac{8}{3}$

H.A.: $R(x) = \dfrac{x^2-2x-15}{3x^2-2x-16}$

$\dfrac{x^2}{3x^2} = \dfrac{1}{3} \Rightarrow \underline{\text{H.A.}: y = \frac{1}{3}}$



8. Use Descarte's Rule of Signs and the Rational Zeros Theorem to find all the real zeros of $f(x) = x^4 - 6x^3 + 7x^2 - 6x - 20$. Use the *real* zeros to factor *f* over the real numbers. This is likely to involve an irreducible quadratic factor.

make sure to use original $f(x)$

$f(x) = x^4 - 6x^3 + 7x^2 - 6x - 20$ 3 or 1 positive zeros

$f(-x) = x^4 + 6x^3 + 7x^2 + 6x - 20$ 1 negative zero

$\frac{p}{q} = \frac{\text{factors of } a_0}{\text{factors of } a_n} = \frac{\pm(1, 2, 4, 5, 10, 20)}{\pm(1)}$

$(x+1)$
$$\begin{array}{r|rrrrr} -1 & 1 & -6 & 7 & -6 & -20 \\ & & -1 & 7 & -14 & 20 \\ \hline & 1 & -7 & 14 & -20 & 0 \end{array} \checkmark$$

normally, check -1 again, but we know there is only one negative zero by Descartes' Rule

$(x^3 - 7x^2 + 14x - 20)(x+1) = f(x)$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 14 & -20 \\ & & 2 & -10 & 8 \\ \hline & 1 & -5 & 4 & \text{Nope} \end{array}$$

$(x-5)$
$$\begin{array}{r|rrrr} 5 & 1 & -7 & 14 & -20 \\ & & 5 & -10 & 20 \\ \hline & 1 & -2 & 4 & 0 \end{array} \checkmark$$

$f(x) = (x^2 - 2x + 4)(x-5)(x+1)$

$\boxed{f(x) = (x+1)(x-5)(x^2 - 2x + 4)}$

This is as far as we can factor over the reals since the discriminant of $x^2 - 2x + 4$ is negative

$a=1 \quad b=-2 \quad c=4$

$b^2 - 4ac$

$\Rightarrow (-2)^2 - 4(1)(4) = 4 - 16 = -12 \Rightarrow$ two non-real solutions that are conjugates of each other.

9. Based on your work in #8 above, find *all* the (real and nonreal) zeros of $f(x) = x^4 - 6x^3 + 7x^2 - 6x - 20$. Use *all* the zeros to write $f(x)$ as the product of *linear* factors. We have to factor $x^2 - 2x + 4$. We already have the discriminant!

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-2) \pm \sqrt{-12}}{2(1)}$

$= \frac{2 \pm \sqrt{(4)(-3)}}{2}$

$x = \frac{2 \pm 2i\sqrt{3}}{2}$

$x = \frac{2(1 \pm i\sqrt{3})}{2}$

$x = 1 \pm i\sqrt{3}$ for $x^2 - 2x + 4$

Now we combine all of $f(x)$ from above

$f(x) = (x+1)(x-5)(x^2 - 2x + 4)$

$\boxed{f(x) = (x+1)(x-5)(x - (1 + i\sqrt{3}))(x - (1 - i\sqrt{3}))}$

conjugate pairs

10. Sketch the graph of $R(x) = \frac{6x^3-7x^2-14x+15}{2x^2-5x+3}$. State the domain, asymptotes, holes, and intercepts. Show them clearly labeled on your graph. (Hint: factor the denominator, then use the zeros of the denominator to check for zeros of the numerator using synthetic division.)

$2x^2-5x+3$

$2x^2-2x-3x+3$

$2x(x-1)-3(x-1)$

$(2x-3)(x-1)$ = denominator

$(x-1)$

$$\begin{array}{r|rrrr} 1 & 6 & -7 & -14 & 15 \\ & & 6 & -1 & -15 \\ \hline & 6 & -1 & -15 & 0 \end{array} \checkmark$$

$R(x)= \frac{(6x^2-x-15)(x-1)}{(2x-3)(x-1)}$

$\mathcal{D} = \{x \mid x \neq 1 \text{ AND } x \neq \frac{3}{2}\}$

$R(x) = \frac{6x^2-x-15}{2x-3}$ with a hole @ $x=1$

$6(-15)=-90$

$-10(9)=-90$ ✓

$-10+9=-1$ ✓

$\frac{6x^2+9x-10x-15}{2x-3}$

$\frac{3x(2x+3)-5(2x+3)}{2x-3}$

$R(x)=\frac{(3x-5)(2x+3)}{2x-3}$

$$\begin{array}{r} 3x+4-\frac{3}{2x-3} \\ 2x-3 \overline{)\,6x^2-x-15} \\ -(6x^2-9x) \\ \hline 8x-15 \\ -(8x-12) \\ \hline -3 \end{array}$$

$y=3x+4$ is O.A.

y-int: $R(0)=\frac{(-5)(3)}{-3}=\frac{-15}{-3}$

$R(0)=5$

y-int: $(0,5)$

4) zeros:

$3x-5=0$

$x=\frac{5}{3}$

$m=1$

$2x+3=0$

$x=-\frac{3}{2}$

$m=1$

5) $-$ 0 $+$ u $-$ 0 $+$ at $-\frac{3}{2}$, $\frac{3}{2}$, $\frac{5}{3}$

test $x=2$: $\frac{(6-5)(4+3)}{4-3}=\frac{12}{1}$ pos.

hmm, knew $R(1)$ is positive from hole that we found...

1) $\mathcal{D}=\mathbb{R}\setminus\{1, \frac{3}{2}\}$

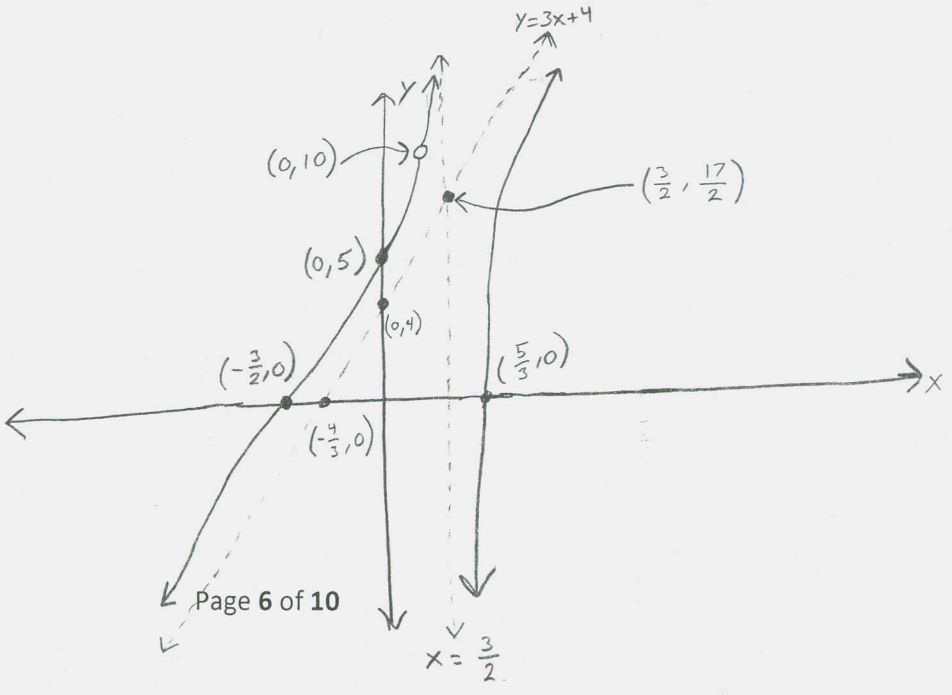
2) hole @ $x=1$

$R(1)=\frac{(3-5)(2+3)}{2-3}=\frac{(-2)(5)}{-1}=10$

hole: $(1,10)$

V.A.: $x=\frac{3}{2}$

3) horizontal asymptote



11. What is the domain of $\sqrt{\dfrac{x-2}{(x+3)^2(x-7)^3}}$?

Need:

$(x+3)^2(x-7)^3 \neq 0$

$(x+3)^2 = 0$ $\quad (x-7)^3 = 0$

$\sqrt{(x+3)^2} = \sqrt{0}$ $\quad \sqrt[3]{(x-7)^3} = \sqrt[3]{0}$

$x+3=0$ $\quad x-7=0$

$x=-3$ $\quad x=7$

So, $x \neq -3$, $x \neq 7$

Also Need:

$\dfrac{x-2}{(x+3)^2(x-7)^3} \geq 0$

$x=2$, $m=1$ $\quad x=-3$, $m=2$

$x=7$, $m=3$

So, $x \in (-\infty, 2] \cup [7, \infty)$

but we cannot include $x=-3$ or $x=7$

$+ \quad + \quad - \quad +$

$-3 \quad 2 \quad 7$

test $x=0$

$\dfrac{-2}{3^2(-7)^3} = \dfrac{-2}{-9(7)^3} =$ positive

thus,

$\boxed{x \in (-\infty, -3) \cup (-3, 2] \cup (7, \infty)}$

12. List each real zero and its multiplicity. Determine whether the graph of $f(x)$ touches or crosses the x-axis at each x-intercept.

$$f(x) = 4x(x^2-4)(x^3+1)^2$$

set each term $= 0$

m = multiplicity

$4x = 0$

$\boxed{x=0,\ m=1,\ \text{cross}}$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$\boxed{x=2,\ m=1,\ \text{cross} \quad x=-2,\ m=1,\ \text{cross}}$

$(x^3+1)^2 = 0$

$\sqrt{(x^3+1)^2} = \sqrt{0}$

$x^3 + 1 = 0$

$x^3 = -1$

$\sqrt[3]{x^3} = \sqrt[3]{-1}$

$x = -1$

$\boxed{x=-1,\ m=2,\ \text{touch}}$

13. Solve each of the following quadratic inequalities. Express your answer interval notation.

a. $x^2 < x + 12$

$x^2 - x - 12 < 0$

$(x-4)(x+3) = 0$

$x-4=0$ $\quad x+3=0$

$x=4$, $m=1$ $\quad x=-3$, $m=1$

$+ \quad - \quad +$

$-3 \quad 4 \quad x$

E.B. $x^2 \rightsquigarrow \uparrow \ldots \nearrow$

want $< 0 \ldots$

$\boxed{x \in (-3, 4)}$

b. $x^2 - x > 11$

E.B.: $x^2 \uparrow \ldots \nearrow$

$x^2 - x - 11 > 0$

$x^2 - x - 11 = 0$

$\left(x^2 - x + \left(-\frac{1}{2}\right)^2\right) - 11 - \frac{1}{4} = 0$

want > 0

$\left(x - \frac{1}{2}\right)^2 - \frac{44}{4} - \frac{1}{4} = 0$

$\left(x - \frac{1}{2}\right)^2 = \frac{45}{4}$

$x - \frac{1}{2} = \pm\sqrt{\frac{45}{4}}$

$x = \frac{1}{2} \pm \frac{\sqrt{9 \cdot 5}}{\sqrt{4}}$

$x = \frac{1}{2} \pm \frac{3\sqrt{5}}{2}$

$+ \quad - \quad +$

$\frac{1-3\sqrt{5}}{2} \quad \frac{1+3\sqrt{5}}{2}$

$\boxed{x \in \left(-\infty, \frac{1-3\sqrt{5}}{2}\right) \cup \left(\frac{1+3\sqrt{5}}{2}, \infty\right)}$

14. Solve the following absolute value inequalities and equations. Draw a quick sketch if it is helpful.

a. $|2x - 2| = 8$

$2x - 2 = 8$ or $2x - 2 = -8$

$2x = 10$ $2x = -6$

$x = 5$ $x = -3$

$x \in \{-3, 5\}$

b. $|2 - 2x| < -8$

NEVAH!

$\boxed{\emptyset}$

An absolute value will never be less than a negative #.

c. $|2x - 2| < 8$

$-8 < 2x - 2 < 8$

$-6 < 2x < 10$

$-3 < x < 5$

$\boxed{x \in (-3, 5)}$

d. $|2 - 2x| > -8$

Always

$\boxed{\mathcal{D} = \mathbb{R}}$

15. Use the Remainder Theorem to find $P(3)$ if $P(x) = 2x^3 + 3x^4 - 5x^2 + 4x - 6$

Synthetic division

$P(x) = 3x^4 + 2x^3 - 5x^2 + 4x - 6$

$$\begin{array}{r|rrrrr} 3 & 3 & 2 & -5 & 4 & -6 \\ & & 9 & 33 & 84 & 264 \\ \hline & 3 & 11 & 28 & 88 & \boxed{258} \end{array}$$

therefore, $\boxed{P(3) = 258}$

16. Use the Intermediate Value Theorem to show that the polynomial function has a zero in the given interval. (Use synthetic division if you want to make Steve really happy!)

$f(x) = 2x^3 + 6x^2 - 8x + 2; \quad [-5, -4]$

$$\begin{array}{r|rrrr} -5 & 2 & 6 & -8 & 2 \\ & & -10 & 20 & -60 \\ \hline & 2 & -4 & 12 & \boxed{-58} \end{array}$$

$f(-5) = -58$

$$\begin{array}{r|rrrr} -4 & 2 & 6 & -8 & 2 \\ & & -8 & 8 & 0 \\ \hline & 2 & -2 & 0 & \boxed{2} \end{array}$$

$f(-4) = 2$

Since $f(x)$ changes sign between $[-5, -4]$, the graph MUST cross the x-axis in that interval (since polynomials are continuous) and thus there IS a zero of $f(x)$ in $x \in [-5, -4]$

17. Find all solutions (both real and non-real) to each equation. Check your answers. $(x^2)^2$

i) $k(x) = \sqrt{3x-5} = 4$ oops!

$k(x) = \sqrt{3x-5} = 4$

$(\sqrt{3x-5})^2 = 4^2$

$3x - 5 = 16$

$3x = 21$

$x = 7$

check: $\sqrt{3(7)-5} \stackrel{?}{=} 4$

$\sqrt{21-5} \stackrel{?}{=} 4$

$\sqrt{16} \stackrel{?}{=} 4$

$4 = 4$ ✓

$\boxed{x \in \{7\}}$

ii) $G(x) = x^4 - 4x^2 - 7 = -2$ let $u = x^2$

$u^2 - 4u - 5 = 0$

$(u-5)(u+1) = 0$

$u - 5 = 0$ or $u + 1 = 0$

$u = 5$ $u = -1$

$x^2 = 5$ $x^2 = -1$

$x = \pm\sqrt{5}$ $x = \pm i$

Check: $x = \pm\sqrt{5}$ $(\sqrt{5})^4 - 4(\sqrt{5})^2 - 5 \stackrel{?}{=} 0$

$25 - 20 - 5 \stackrel{?}{=} 0$

$25 - 25 = 0$

$0 = 0$ ✓

check: $x = \pm i$

$(\sqrt{-1})^4 - 4(\sqrt{-1})^2 - 5 \stackrel{?}{=} 0$

$(-1)^2 - 4(-1) - 5 \stackrel{?}{=} 0$

$1 + 4 - 5 = 0$

$5 - 5 = 0$

$0 = 0$ ✓

$\boxed{x \in \{\pm\sqrt{5}, \pm i\}}$

$i = \sqrt{-1}$

iii) $(u^2 + 2u)^2 - 2(u^2 + 2u) = 3$

Should have been a square here! sorry!

let $w = (u^2 + 2u)$

$w^2 - 2w - 3 = 0$

$(w-3)(w+1) = 0$

$w - 3 = 0$ or $w + 1 = 0$

$(u^2 + 2u) - 3 = 0$ $w = -1$

$(u+3)(u-1) = 0$ $u^2 + 2u = -1$

$u = -3, u = 1$ $u^2 + 2u + 1 = 0$

$(u+1)^2 = 0$

$u + 1 = 0$

$u = -1$

$\boxed{x \in \{-3, -1, 1\}}$

Check:

$u = -3$:

$((-3)^2 + 2(-3))^2 - 2((-3)^2 + 2(-3)) - 3 \stackrel{?}{=} 0$

$(9-6)^2 - 2(9-6) - 3 \stackrel{?}{=} 0$

$9 - 6 - 3 = 0$

$9 - 9 = 0$

$0 = 0$ ✓

Checked ~~x~~ $u = 1$ and $u = -1$ with calculator.

iv) $h(x) = \sqrt{x+40} - \sqrt{x} = 4$

$(\sqrt{x+40} - \sqrt{x})^2 = 4^2$

$x + 40 - 2\sqrt{x}\sqrt{x+40} + x = 16$

$2x + 24 = 2\sqrt{x}\sqrt{x+40}$

$(x+12)^2 = (\sqrt{x}\sqrt{x+40})^2$

$x^2 + 24x + 144 = x(x+40)$

$\cancel{x^2} + 24x + 144 = \cancel{x^2} + 40x$

$144 = 16x$

$x = \frac{144}{16}$

$x = 9$

put all $\sqrt{\ }$ terms on one side then square each side

divided both sides by 2

same thing here

Now just solve for x

check:

$\sqrt{9+40} - \sqrt{9} \stackrel{?}{=} 4$

$\sqrt{49} - 3 \stackrel{?}{=} 4$

$7 - 3 = 4$

$4 = 4$ ✓

$\boxed{x \in \{9\}}$

18. Put the following function into the form $a(x-h)^2+k$ and graph it. State the domain and range of the function.

$y = x^2 - 6x + 11$

complete the square... keep all terms on the same side of equation. Don't set = 0!

$y = (x^2 - 6x + (-3)^2) + 11 - 9$

$(\frac{1}{2} \cdot b)^2 = (\frac{1}{2} \cdot -6)^2 = (\frac{-6}{2})^2 = (-3)^2 = 9$

$y = (x-3)^2 + 2$

that was easy!

$a = 1, \quad h = 3, \quad k = 2$

vertex: $(h,k) \in (3,2)$

x-int: $y = (x-3)^2 + 2 = 0$

$(x-3)^2 = -2$

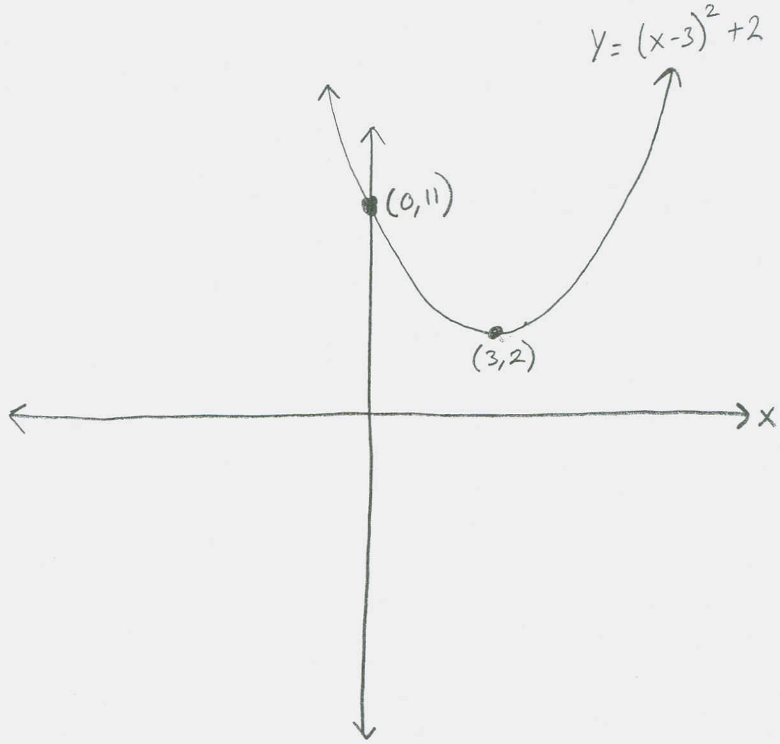
No real zeros!

the graph does not touch the x-axis.

y-int: $y(0) = (0-3)^2 + 2$

$= 9 + 2 = 11$

$(0,11)$ is y-int.



$\mathcal{D} = \mathbb{R} = (-\infty, \infty)$

$\mathcal{R} = [2, \infty)$