§ 5.1, 5.2 Systems of Linear Equations

Substitution

Elimination → Gaussian - Back-substitution w/ w/o matrices

Elimination → Gauss-Jordan - Back-elimination

Write much, think little.

$$\begin{aligned} x + y + z &= 6 \\ 2x - 2y - z &= -5 \\ 3x + y - z &= 2 \end{aligned} \longrightarrow \begin{aligned} x + y + z &= 6 \\ ay + bz &= c \\ dy + ez &= f \end{aligned}$$

Aiming for triangular system

$$\begin{aligned} x + y + z &= 6 \\ y + az &= b \\ z &= c \end{aligned}$$

Now, Gauss says use $z = c$ to back-substitute to the solution.

E1 $x + y + z = 6$
E2 $2x - 2y - z = -5$
E3 $3x + y - z = 2$

E1
$-2E1 + E2$
$-3E1 + E3$

$x + y + z = 6$
$-4y - 3z = -17$
$y + 2z = 8$

Scratch: $-2E1$ $-2x - 2y - 2z = -12$
E2 $2x - 2y - z = -5$
$-2E1+E2$ $-4y - 3z = -17$

$-3E1$ $-3x - 3y - 3z = -18$
E3 $3x + y - z = 2$
$-3E1+E3$ $-2y - 4z = -16$
$x -\frac{1}{2}$ $y + 2z = 8$

$x + y + z = 6$
E2⇔E3 $y + 2z = 8$
$-4y - 3z = -17$

$4E2 + E3$

$x + y + z = 6$
$y + 2z = 8$
$z = 3$

Scratch:
4E2 $4y + 8z = 32$
E3 $-4y - 3z = -17$
$5z = 15$
$z = 3$

$\boxed{z = 3}$
$y + 2z = y + 2(3) = y + 6 = 8$
$\boxed{y = 2}$
$x + y + z = x + 2 + 3 = x + 5 = 6$
$\boxed{x = 1}$

This is 3 planes intersecting at the point $(x, y, z) = (1, 2, 3)$

Solution Set: $\{(1,2,3)\}$

Nov 7-7:52 AM

3 things can happen when trying to solve a system.

1. Unique Solution - A single point Independent
2. No Solution — Inconsistent.
3. Infinitely Many solutions — Dependent.

Re-do previous with matrices!

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -2 & -1 & -5 \\ 3 & 1 & -1 & 2 \end{array}\right] \begin{array}{r} E1 \\ -2E1+E2 \\ -3E1+E3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & -3 & -17 \\ 0 & -2 & -4 & -16 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{E3}{-2} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & -3 & -17 \\ 0 & 1 & 2 & 8 \end{array}\right] E1 \leftrightarrow E2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & -4 & -3 & -17 \end{array}\right]$$

$$\begin{array}{r} E1 \\ E2 \\ 4E2+E3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 5 & 15 \end{array}\right] \frac{1}{5}E2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$z = 3$, etc.

What if there isn't a unique solution?

$E1 \quad y + z = 5 \qquad E1 \quad y + z = 5$

$E2 \quad 3y + 3z = 15 \qquad -3E1 + E2 \quad \boxed{0 = 0}$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & 3 & 15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} y + z = 5 \\ y = 5 - z \end{array}$$

$\{ (y, z) \mid (5-z, z)$, where z is real $\}$

$$\begin{array}{l} x - y + z = 7 \\ y + z = 5 \\ 3y + 3z = 15 \end{array} \xrightarrow{\text{SAME step}} \begin{array}{l} x - y + z = 7 \\ y + z = 5 \\ 0 = 0 \end{array}$$

So z is free! $\boxed{y = 5 - z}$, so

$x - y + z = x - (5 - z) + z = x - 5 + 2z = 7$

$x + 2z = 12$

$\boxed{x = 12 - 2z}$

Solution set $\{ (12 - 2z, 5 - z, z) \mid z \text{ is real} \}$

$\boxed{\text{One free variable.}}$

one-dimensional solution set

Solution set is a line in 3-space.

Another representation:

$\{ (x, y, z) \mid x = 12 - 2z, y = 5 - z, z \in \mathbb{R} \}$

Dependent & consistent

$$\begin{aligned} x - y + z &= 7 \\ y + z &= 5 \\ 3y + 3z &= 7 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x - y + z &= 7 \\ y + z &= 5 \\ 0 &= -8 \quad !? \end{aligned}$$

FALSE

This shows that there is no solution:

Our system-solving technique rests on the assumption that there IS a solution. Reasoning correctly from a faulty premise leads to the absurdity, $0 = -8$.

The faulty premise: "There is a solution"

So, There must NOT be a solution.

$z = \mathbb{R}$, $z =$ all reals BAD

$z \in \mathbb{R}$, z is any real # GOOD

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§ 5.3 other systems

we use substitution, exclusively.

$$5x - y = 6$$
$$y = x^2$$

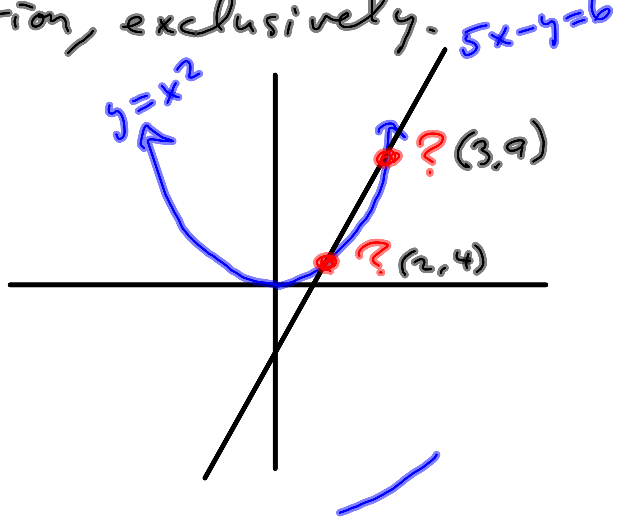
x	y
0	-6
$6/5$	0

$$5x - y = 5x - x^2 = 6$$

$$-x^2 + 5x - 6 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x-3)(x-2) = 0$$

$x = 2,\ x = 3$

$y = 4,\ y = 9$ from $y = x^2$



Solution Set:

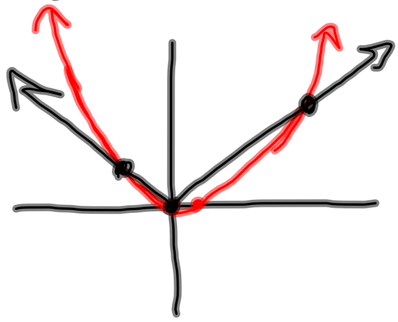
$\{(2,4), (3,9)\}$

$$\left.\begin{aligned} y &= 18x^2 - x \\ y &= x \end{aligned}\right\} \quad x = 18x^2 - x$$

$y = |6x|$

$\boxed{y = x + 35}$

$18x^2 - x = x(18x - 1)$



$|6x| = x + 35$

$6x = x + 35$ or $6x = -x - 35$

$5x = 35$ $\quad$ $7x = -35$

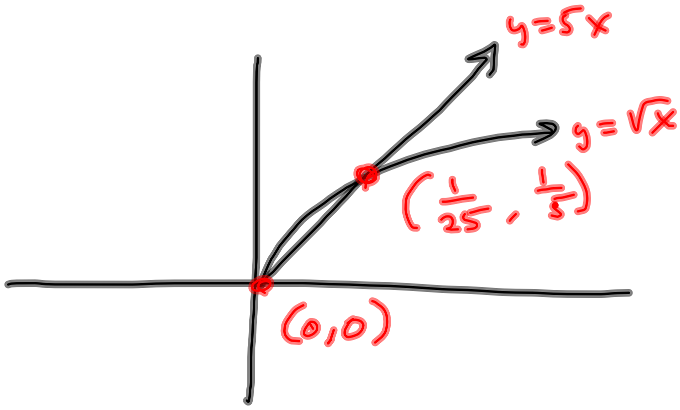
$x = 7$ or $x = -5$

$y = 42$ $\quad$ $y = 30$

$\{(7, 42), (-5, 30)\}$

$-5 + 35 = 30$

$$y = \sqrt{x}$$
$$y = 5x$$



$$5x = \sqrt{x}$$
$$(5x)^2 = (\sqrt{x})^2$$
$$25x^2 = x$$
$$25x^2 - x = 0$$
$$x(25x-1) = 0$$
$$x=0 \text{ OR } x = \frac{1}{25}$$

$y = x^3 - 4x$

$y = x$

$x = x^3 - 4x = x$

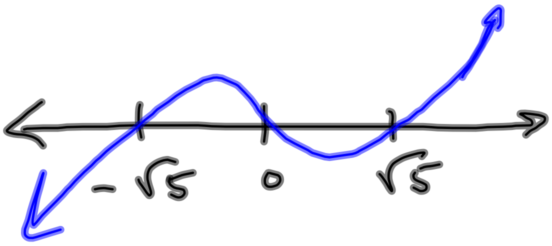
$x^3 - 5x = 0$

$x(x^2 - 5) = 0$

$x = 0$ or $x^2 - 5 = 0$

$x^2 = 5$

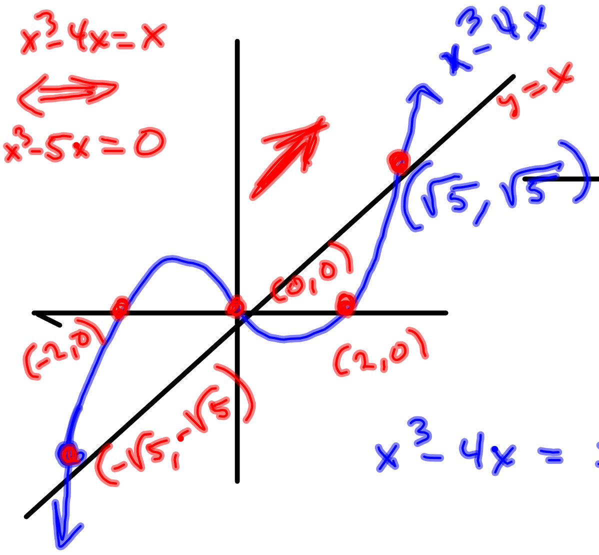
$x = \pm\sqrt{5}$

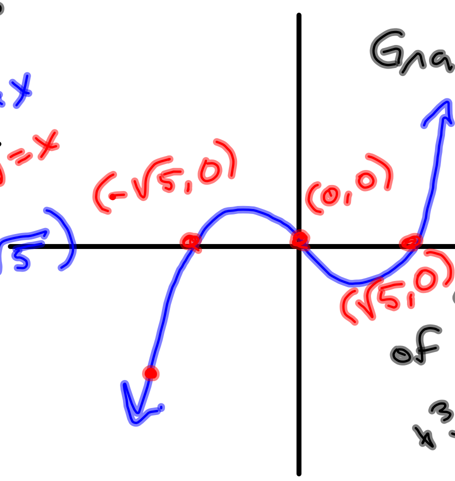


$x^3 - 4x = x$

$\Longleftrightarrow$

$x^3 - 5x = 0$





Graph of $x^3 - 5x$

The zeros of $x^3 - 5x$ are the roots of the equation $x^3 - 4x = x$

$x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$