§3.2

Rational Zeros

Synthetic Division

Fundamental Theorem of Algebra - Every polynomial has at least one zero.

Leads to $a_n(\underline{x-r_1})(x-r_2)\cdots(x-r_n)$

We can split any polynomial into linear factors. We can always find n zeros.

Factor Theorem - Linear Factors correspond to zeros.

Remainder Theorem - Divide by $x-c$ to find $f(c)$

§3.2 Questions? $3+2i$

#67 $P(x) = x^4 + 2x^3 - x^2 - 4x - 2$

Find all real & nonreal zeros for $P(x)$

→ Are pure imaginary in this stretch of exercises.

$a_n = 1 \rightsquigarrow q$'s

$a_0 = -2 \rightsquigarrow p$'s

Any zero of the form $\frac{p}{q}$ will be such that p is a factor of -2 (a_0) & q is a factor of 1 (a_n)

$3x^5 - 7$ $\pm 1, \pm 7, \pm \frac{1}{3}, \pm \frac{7}{3}$

$\frac{p}{q}$: $\pm 1, \pm 2$

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & -1 & -4 & -2 \\ & & 1 & 3 & 2 & -2 \\ \hline & 1 & 3 & 2 & -2 & \end{array}$$
Nope

$$\begin{array}{r|rrrrr} -1 & 1 & 2 & -1 & -4 & -2 \\ & & -1 & -1 & 2 & 2 \\ \hline -1 & 1 & 1 & -2 & -2 & 0 \\ & & -1 & 0 & 2 & \\ \hline & 1 & 0 & -2 & 0 & \\ & x^2 & x & c & r & \end{array}$$

Yes! Yes!

Try $x=-1$ again:

$x=-1$ is a zero. we split off $x+1$:

$P(x) = (x+1)(x^3 + x^2 - 2x - 2)$

This says $P(x) = (x+1)(x+1)(x^2-2)$

Now, to finish, we find zeros of x^2-2

$x^2 - 2 = 0$
$x^2 = 2$
$x = \pm\sqrt{2}$

$a=1, b=0, c=-2$
$b^2 - 4ac = 0 - 4(1)(-2) = 8$
$x = \frac{-0 \pm \sqrt{8}}{2} = \frac{\pm\sqrt{8}}{2} = \frac{\pm 2\sqrt{2}}{2} = \pm\sqrt{2}$

$x^2 - 2 =$
$x^2 - \sqrt{2}^2 =$
$(x-\sqrt{2})(x+\sqrt{2}) = 0$
$\Rightarrow x = \pm\sqrt{2}$

Punchline: zeros are $-1, \pm\sqrt{2}$ 3.2

3.3 we point out that $x=-1$ worked twice by saying $x=-1$ is of multiplicity 2.

3.3 wants to see factored form

$P(x) = (x+1)^2(x-\sqrt{2})(x+\sqrt{2})$ is now "Split into linear factors."

Factor Theorem
Rational Zeros
Synthetic Division

---

§3.3 Descartes' Rule of signs
Conjugate Pairs Theorem → "Imaginary won't cut it." Hence forth, "Nonreal" is the term → $2+i\sqrt{3}$

Bounds on Real Zeros

P(x) in factored form:

$(x-2)^2(x+5)^3(x-7)^1$

$x=2$ is a zero of multiplicity 2
$x=-5$ " " " " " 3
$x=7$ " " " " " 1

P(x) has 6 zeros. We count the multiplicities.

Descartes':

$f(x) = \underbrace{x^3 - 7x^2}_{1}\underbrace{\;-7x^2+5x}_{2}\underbrace{+5x-11}_{3}$ sign changes.

Either 3 or 1 positive real zeros.

$f(-x) = (-x)^3 - 7(-x)^2 + 5(-x) - 11$

$= -x^3 - 7x^2 - 5x - 11$

No sign changes.

No negative zeros.

Can save time on the search.

#67 $P(x) = x^4 + 2\underbrace{x^3 - x^2}_{} - 4x - 2$

1 sign change

$\Longrightarrow$ 1 positive real zero.

$P(-x) = \underbrace{x^4 - 2x^3}_{1} \underbrace{- x^2 + 4x}_{2} \underbrace{- 2}_{3}$

Either 3 or 1 negative real zeros.

Turned out we had

$x = -1$ (twice) } 3 negative

$x = -\sqrt{2}$

$x = \sqrt{2}$ ——— 1 positive.

Conjugate pairs Theorem

If $a+bi$ is a zero of $f(x)$ and $f(x)$ has all real coefficients, then $a-bi$ is also a zero.

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$f(x) = x^2 - 2x + 5 \overset{\text{SET}}{=} 0$

$\Longrightarrow x^2 - 2x = -5$

$x^2 - 2x + 1^2 = -5 + 1$

$(x-1)^2 = -4$

$x - 1 = \pm\sqrt{-4} = \pm i\sqrt{4} = \pm 2i$

$x = 1 \pm 2i$

$x = \underline{1+2i}$ is a zero $\Longrightarrow x = \underline{1-2i}$ is, too.

Conjugate pairs.

Find all zeros of $2x^3-5x^2-6x+4=f(x)$

$a_n=2$ $\frac{p}{q}$: $\pm1, \pm2, \pm4, \pm\frac{1}{2}$ Rat'l

$a_0=4$

Descartes: 2 or 0 positive real zeros

$f(-x)=-2x^3-5x^2+6x+4$

1 negative real zero.

$$\begin{array}{r|rrrr} -1 & 2 & -5 & -6 & 4 \\ & & -2 & 7 & \\ \hline & 2 & -7 & 1 & \text{No} \end{array}$$

$$\begin{array}{r|rrrr} -2 & 2 & -5 & -6 & 4 \\ & & -4 & 18 & -24 \\ \hline & 2 & -9 & 12 & \text{No.} \end{array}$$

$$\begin{array}{r|rrrr} -4 & 2 & -5 & -6 & 4 \\ & & -8 & 52 & \\ \hline & 2 & -13 & & \text{No} \end{array}$$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -5 & -6 & 4 \\ & & -1 & 3 & \frac{3}{2} \\ \hline & 2 & -6 & -3 & \text{Nope.} \end{array}$$

So, whatever the negative zero is, it isn't rational. Must be a $\sqrt{}$ in it.

So, look for POSITIVE rational zeros.

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & -6 & 4 \\ & & 1 & -2 & -4 \\ \hline & 2 & -4 & -8 & 0 \end{array}$$

This says

$f(x) = (x-\frac{1}{2})(2x^2-4x-8)$

$= (x-\frac{1}{2})(2)(x^2-2x-4)$

$= 2(x-\frac{1}{2})(x^2-2x-4)$

$x^2-2x-4 \overset{\text{SET}}{=} 0$

$x^2-2x = 4$

$x^2-2x+1^2 = 4+1$

$(x-1)^2 = 5$

$x-1 = \pm\sqrt{5}$

$x = 1 \pm \sqrt{5}$

Zeros: $x = \frac{1}{2}, 1 \pm \sqrt{5}$

Factored Form:

$2(x-\frac{1}{2})(x-(1+\sqrt{5}))(x-(1-\sqrt{5}))$

$= a_n(x-r_1)(x-r_2)(x-r_3)$

$x=c$ is a positive bound on real zeros
if, when you divide $P(x)$ by $x-c$, the bottom
row is nonnegative #s.

$x=c$ is a negative bound on real zeros
if, when you divide $P(x)$ by $x-c$, the bottom
row alternates in sign.

$$(x+2)(x^2-2x+1) = \begin{array}{r} x^3-2x^2+x \\ 2x^2-4x+2 \\ \hline x^3-3x+2 \end{array}$$

positive

Claim: $x=3$ is a bound on real zeros.

$$\begin{array}{r|rrrr} 3 & 1 & 0 & -3 & 2 \\ & & 9 & 27 & 72 \\ \hline & 3 & 9 & 24 & 74 \end{array}$$

Claim: $x=-3$ is a negative lower bound on zeros

$$\begin{array}{r|rrrr} -3 & 1 & 0 & -3 & 2 \\ & & -3 & 9 & -18 \\ \hline & 1 & -3 & 6 & -16 \end{array}$$

§3.3 #s 11, 12, 15, 17, 20, 23, 31, 37,
43, 46, 47, 61, 65, 69

66 Solve.

$x^4 - 4x^3 + 7x^2 - 16x + 12 = 0$

$a_n = 1$ $\frac{p}{q}$: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$a_0 = 12$

Decartes: 4, 2, or 0 positive roots.

$f(-x) = x^4 + 4x^3 + 7x^2 + 16x + 12$ No negative roots

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 7 & -16 & 12 \\ & & 1 & -3 & 4 & -12 \\ \hline 1 & 1 & -3 & 4 & -12 & 0 \\ & & 1 & -2 & & \\ \hline & 1 & -2 & 2 & & \end{array}$$

$(x-1)\underline{(x^3 - 3x^2 + 4x - 12)}$

Nope.

$$\begin{array}{r|rrrr} 2 & 1 & -3 & 4 & -12 \\ & & 2 & -2 & \\ \hline & 1 & -1 & 2 & \end{array}$$

Nope

$$\begin{array}{r|rrrr} 4 & 1 & -3 & 4 & -12 \\ & & 4 & 4 & \\ \hline & 1 & 1 & 8 & \end{array}$$

Nop

$i^2 = -1$

$\sqrt{-1} = i$

Forgot 3

$$\begin{array}{r|rrrr} 3 & 1 & -3 & 4 & -12 \\ & & 3 & 0 & 12 \\ \hline & 1 & 0 & 4 & \end{array}$$

Sweet!

$(x-1)(x-3)(x^2+4)$

$x^2 + 4 = (x - 2i)(x + 2i) = x^2 - (2i)^2$

$x^2 = -4$

$x = \pm\sqrt{-4} = \pm 2i$

$x = 1, 3, \pm 2i$

$(x-1)(x-3)(x-2i)(x+2i)$